CARDIFF UNIVERSITY
EXAMINATION PAPER

Academic Year: 2006/2007

Examination Period: Spring

Examination Paper Number: CM0167Solutions

Examination Paper Title: Mathematics for Computer Science

SOLUTIONS

Duration: 2 hours

Do not turn this page over until instructed to do so by the Senior Invigilator.

Structure of Examination Paper:

There are 22 pages.
There are 9 questions in total.
There are no appendices.
The mark obtainable for a question or part of a question is shown in brackets alongside the question.

Students to be provided with:

The following items of stationery are to be provided:
ONE answer book.

Instructions to Students:

Answer all questions.
The use calculators **without** programmable memory is permitted.
The use of translation dictionaries between English or Welsh and a foreign language bearing an appropriate departmental stamp is permitted in this examination.

Q1. Apply the binary tree sort *algorithm to sort the following data*

$$8 \quad 2 \quad 9 \quad 12 \quad 6 \quad 4 \quad 5 \quad 1 \quad 9$$

and represent it with a binary tree.

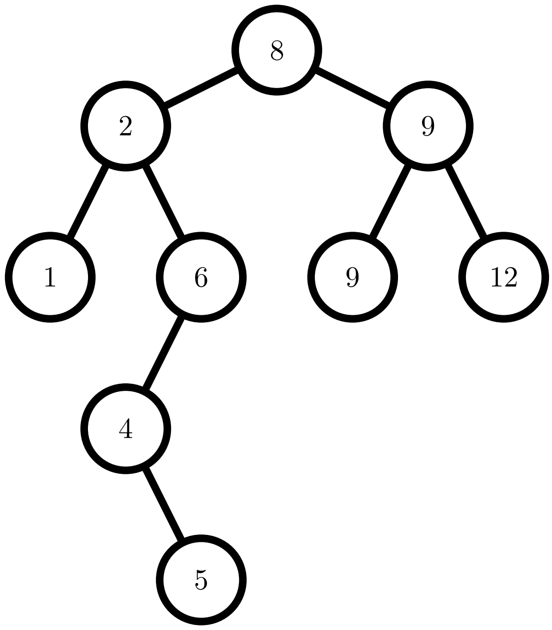
How would you use the tree to sort the data in ascending **and** descending *order?*

Recap: *listing of algorithm not required for solution* The binary tree sort algorithm:

Create tree via:

- First number is the root.
- Put number in tree by traversing to an end vertex
  - If number less than or equal vertex number go left branch
  - If number greater than vertex number go right branch

So Binary Tree for above data is:



4 Marks for correct sort tree

To sort data: Traverse tree in top down left most branch first order for ascending order Traverse tree in top down right most branch first order for descending order

1 Mark Each for sort order

6 Marks Bookwork Total — Unseen problem

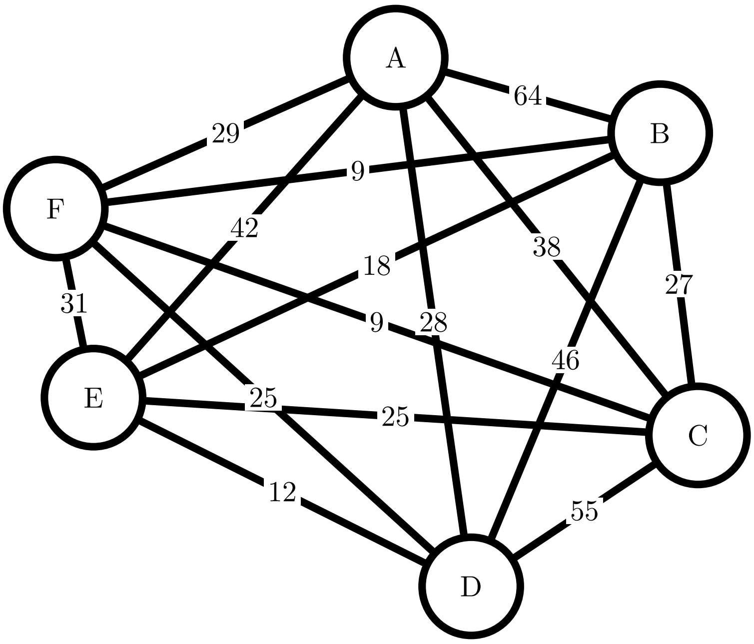
Q2. Consider the following table of distances between the cities A, B, C, D, E and F:

	A	B	C	D	E	F
A	–	64	38	28	42	29
B	64	–	27	46	18	9
C	38	27	–	55	25	9
D	28	46	55	–	12	25
E	42	18	25	12	–	31
F	29	9	9	25	31	–

(a) Find an upper bound for the solution to the travelling salesman problem for the six cities above using the heuristic nearest neighbour *algorithm.*

(b) Find a lower bound for the solution to the travelling salesman problem by removing city A. *[9]*

Graph for above data is:



(a) Upper Bound Solution

Recap: *listing of algorithm not required for solution* To get an upper bound we use the following algorithm (The heuristic/nearest neighbour algorithm)

The idea for the heuristic algorithm is similar to the idea of Prim's algorithm, except that we build up a cycle rather than a tree.

- START with all the vertices of a complete weighted graph.

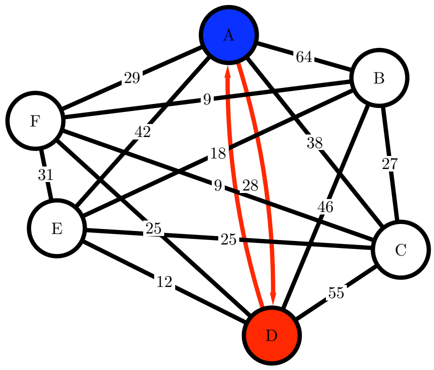
- Step 1: Choose any vertex and find a vertex joined to it by an edge of minimum weight. Draw these two vertices and join them with two edges to form a cycle. Give the cycle a clockwise rotation.

- Step 2: Find a vertex not currently drawn, joined by an edge of least weight to a vertex already drawn. Insert this new vertex into the cycle in front of the 'nearest' already connected vertex.

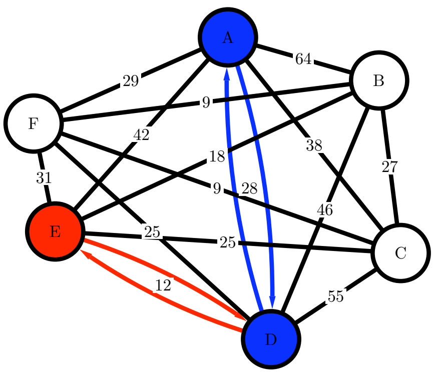
- REPEAT Step 2 until all the vertices are joined by a cycle, then STOP.

The total weight of the resulting Hamiltonian cycle is then an upper bound for the solution to the travelling salesperson problem.

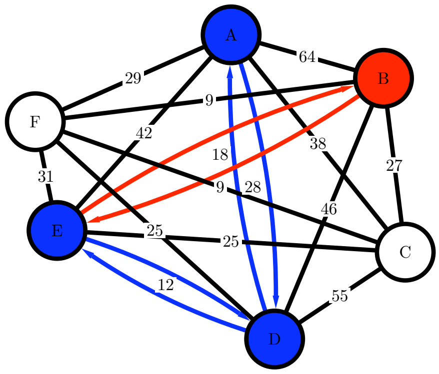
Step 1: Choose Vertex A (*if other vertices chosen a valid but different answer possible*). Draw A. Lowest weight is AD. So Draw D and draw AD as a clockwise cycle.



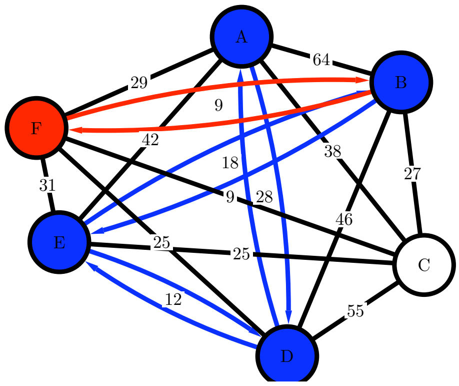
Step 2: Vertices Drawn: A, D. Lowest Weight from undrawn vertex to drawn vertex is DE. So Draw E and draw DE as a clockwise cycle.



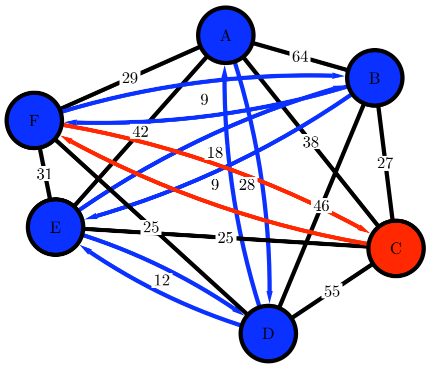
Step 3: Vertices Drawn: A, D, E. Lowest Weight from undrawn vertex to drawn vertex is EB. So Draw B and draw EB as a clockwise cycle.



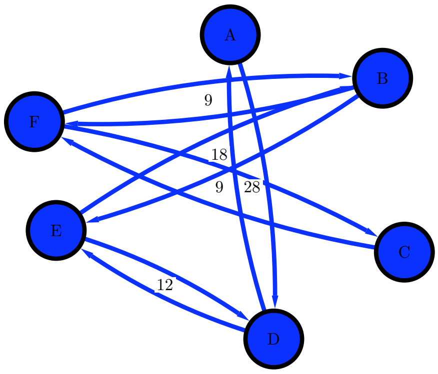
Step 4: Vertices Drawn: A, D, E, B. Lowest Weight from undrawn vertex to drawn vertex is BF. So Draw F and draw BF as a clockwise cycle.



Step 5: Vertices Drawn: A, D, E, B, F. Lowest Weight from undrawn vertex to drawn vertex is FC. So Draw C and draw FC as a clockwise cycle.



So Hamiltonian Cycle create is:



The **Upper BOUND** for the TSP of this problem is the weight of this cycle which is:

$$2 \times (28 + 12 + 18 + 9 + 9) = 152$$

1 mark for each step plus 1 mark for cycle plus 2 marks for Upper bound calculation

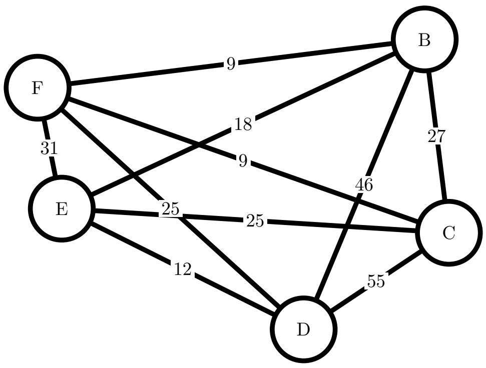
8 Marks total — unseen problem

(b) Lower Bound Solution

Recap: *listing of algorithm not required for solution* Lower bound for the travelling salesperson problem algorithm:

- Step 1: Choose a vertex V and remove it from the graph.
- Step 2: Find a minimum spanning tree connecting the remaining vertices, and calculate its total weight w.
- Step 3: Find the two smallest weights, w_1 and w_2, of edges incident with V.
- Step 4: Calculate the lower bound $w + w_1 + w_2$.

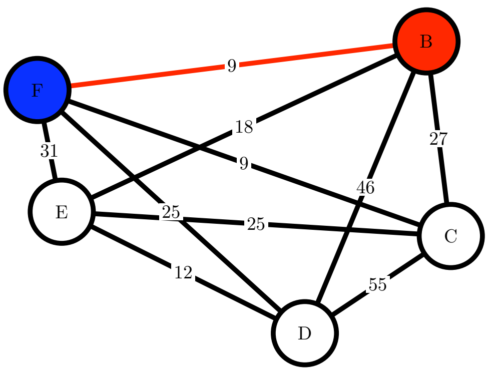
Step 1: Choose Vertex A. Remove A from the graph:



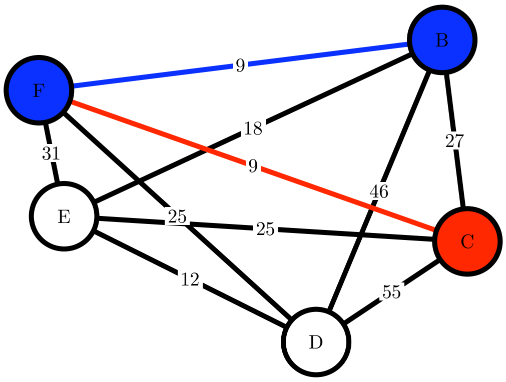
Step 2: Use Prim's Algorithm to find minimum spanning tree:

- START with all the vertices of a weighted graph.
- Step 1: Choose and draw any vertex.
- Step 2: Find the edge of least weight joining a drawn vertex to a vertex not currently drawn. Draw this weighted edge and the corresponding new vertex .
- REPEAT Step 2 until all the vertices are connected, then STOP.

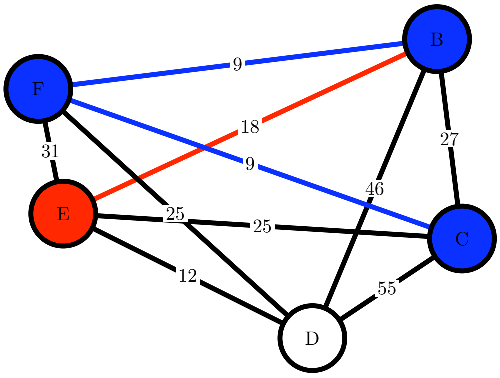
Step 2.1: Choose and draw F. Edge of least weight to F is either B or C. Choose B. Draw B and edge FB:



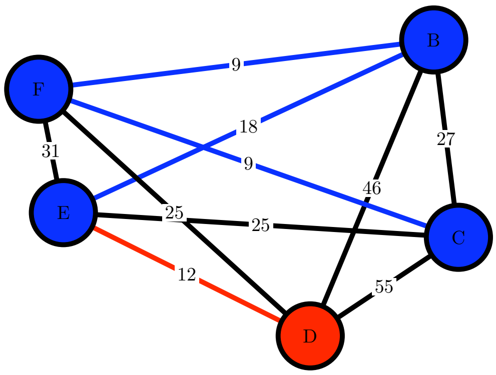
Step 2.2: Vertices drawn: F, B . Edge of least weight from a non-drawn vertex to a drawn vertex is FC Choose C. Draw C and edge FC:



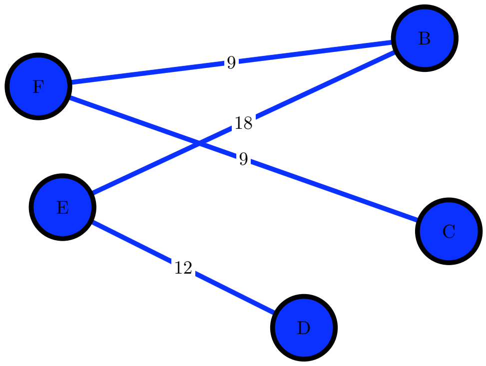
Step 2.3: Vertices drawn: F, B, C. Edge of least weight from a non-drawn vertex to a drawn vertex is BE Choose E. Draw E and edge BE:



Step 2.3: Vertices drawn: F, B, C, E. Edge of least weight from a non-drawn vertex to a drawn vertex is ED Choose D. Draw D and edge ED:



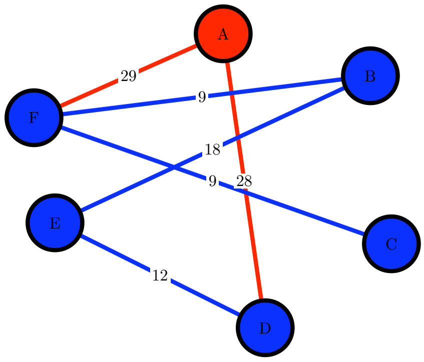
Step 2.4: So minimum spanning tree is:



The weight of this tree is:

$$w = 9 + 9 + 18 + 12 = 48$$

Step 3 : Now add the two least weighted edges to A which are AD with weight $w_1 = 28$ and AF with weight $w_2 = 29$



So the **lower bound** of this TSP problem is:

$$w + w_1 + w_2 = 48 + 28 + 29 = 105$$

1 mark for each step plus 1 mark for cycle graph plus 1 mark for lower bound calculation

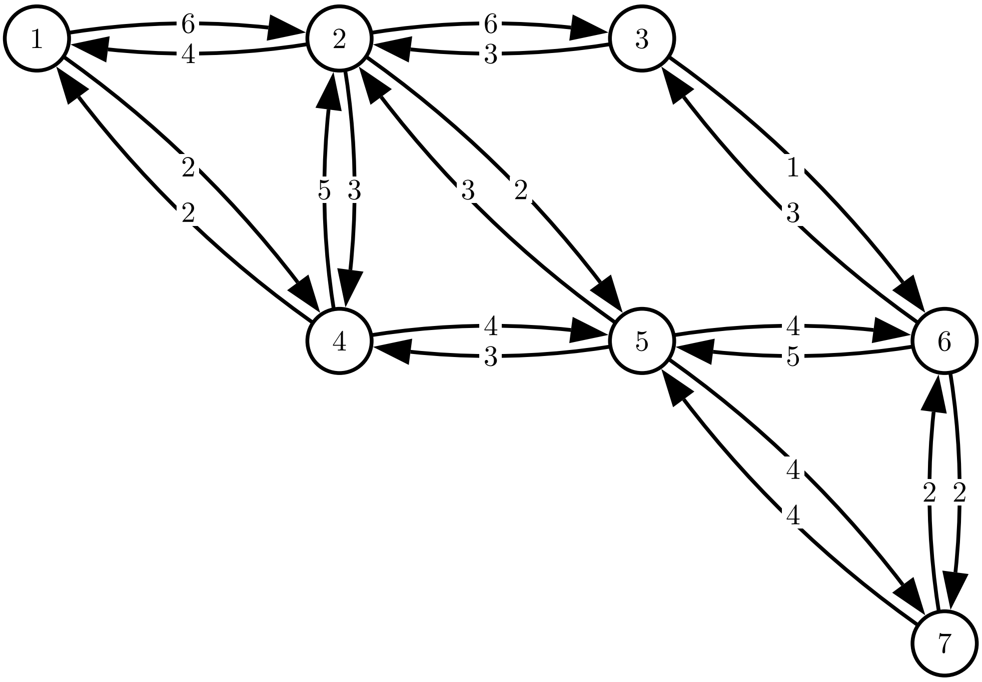
9 Marks total — unseen problem

Q3. Consider the following table of average capacities of communication links in a computer network:

Vertices	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1	–	6	–	2	–	–	–
2	4	–	6	3	2	–	–
3	–	3	–	–	–	1	–
4	2	5	–	–	4	–	–
5	–	3	–	3	–	4	4
6	–	–	3	–	5	–	2
7	–	–	–	–	4	2	–

(a) Represent the above table as digraph *of the computer network?*

The digraph is as follows:



4 Marks — Unseen Problem

(b) Using Djikstra's algorithm*, Find the shortest path from vertex 1 to all other vertices. Express your solution as a shortest path tree.*

Recap: **Djikstra's algorithm** *listing of algorithm not required for solution*

Aim: Finding the **shortest path** from a vertex S to a vertex T in a weighted digraph.

- START: Assign potential 0 to S.
- **General Step**:
  1. Consider the vertex (or vertices) just assigned a potential.
  2. For each such vertex V, consider each vertex W that can be reached from V along an arc VW and assign W the label
  `potential of` $V +$ `distance` VW
  unless W already has a smaller or equal label assigned from an earlier iteration.
  3. When all such vertices W have been labelled, choose the smallest vertex label that is not already a potential and make it a potential at each vertex where it occurs.
- REPEAT the general step with the new potentials.
- STOP when T has been assigned a potential.

The *shortest distance* from S to T is the *potential* of T.

To find the *shortest path*, trace backwards from T and include an arc VW whenever we have

`potential of` W - `potential of` $V =$ `distance` VW

until S is reached.

Applying Dijkstra's algorithm for vertex 1, we get the following table:

Step	Vertex to be marked	Current potential	Distance to Vertex 1	2	3	4	5	6	7	Unchosen vertices
Step 1	1	0	0	6	–	**2**	–	–	–	2,3,4,5,6,7
Step 2	4	2	0	**6**	–	**2**	6	–	–	2,3,5,6,7
Step 3	2	6	0	**6**	12	**2**	6	–	–	3,5,6,7
Step 4	5	6	0	**6**	12	**2**	**6**	**10**	10	3,6,7
Step 5	6	10	0	**6**	12	**2**	**6**	**10**	**10**	3,7
Step 6	7	10	0	**6**	**12**	**2**	**6**	**10**	**10**	3

This is explained as follows (**= choose**, **= chosen**, **= dont overwrite**):

Step 1 — Only Valid Paths are to vertices 2 and 4. Path to **4** is clearly lowest.

Step 2 — Update Current Potential (2). 4 can link to 2 but this potential, 7, is **more** than current potential 6 **so don't replace**. Add a potential to Vertex 5. Vertex **2** has joint lowest weight, 6, with vertex 5. Let choose it Vertex **2**.

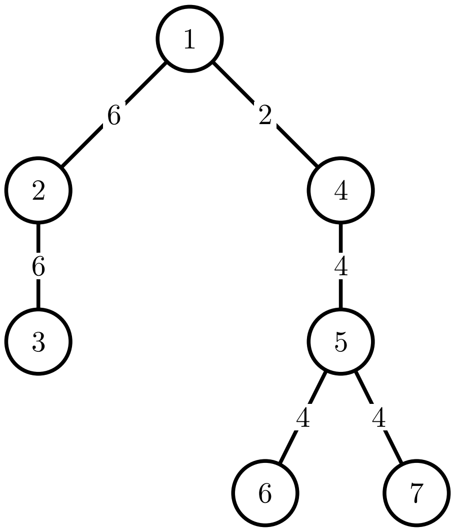
Step 3 — Update Current Potential (6). Can add path to vertex 3 and update vertex 5. But Vertex 5 has lower weight, 6, from before. This weight to is lowest so choose vertex **5**.

Step 4 — Update Current Potential (6). Can add paths to vertex 6 and vertex 7. Vertex 6 is now the joint lowest with vertex 7. Choose vertex **6**

Step 5 — Update Current Potential (10). Can't add a path from vertex 6 to vertex 3 as its weight, 13, higher than current weight, 12. Also path from vertex 6 to vertex 7 **weight, 12,** is more than current weight, 10, so don't update. Choose vertex **7** as weight is lowest.

Step 6 — Update Current Potential (10). Cant update any values paths as vertices already marked. Only **vertex 3 left to choose**.

We can therefore construct the following tree, by following back shortest paths from each vertex to vertex according to Dijkstra's algorithm, to show how we go from vertex one to other nodes:



Shortest paths noted on the edges of the above graph.

6 Marks, 1 for each step in the table

3 Marks for the tree and shortest paths

9 Marks Total — Unseen Problem

(c) Write down the *routing table* for vertex 1.

Therefore we can construct a **routing table** for vertex 1 as follows:

Destination	2	3	4	5	6	7
Next Node	2	4	4	4	4	4

2 Marks Total — Unseen Problem

Q4. Three bags contain red and white balls. Bag 1 contains 8 red and 2 white balls, bag 2 contains 3 red and 4 white balls and bag 3 contains 1 red and 6 white balls.

A person wishes the draw a single ball:

(a) *What is the* probability *that a red ball is drawn at random if all the bags' balls are mixed together?*

The number of respective balls in bag 1 is 10, bag 2 is 7 and bag 3 is 7.
The total number of balls is **24**.
The number of respective **red** balls in bag 1 is 8, bag 2 is 3 and bag 3 is 1.
The total number of **red** balls is **12**.
Therfore the probability of choosing a red ball from all the balls mixed together is:

$$P(R) = \frac{\text{number of } \textbf{red} \text{ balls}}{\text{number of all balls}} = \frac{12}{24} = 0.5$$

3 marks unseen problem

(b) *What is the* probability *that a red ball is picked when any one of the bags is first selected at random?*

The probability of choosing any bag_i, $P(B_i) = \frac{1}{3}$ for $i = 1, 2, 3$
By *Theorem of Total Probabilty*:

$$P(R) = \Sigma_{i=1}^{3} P(B_i)P(R|B_i)$$

Now $P(R|B_1) = \frac{8}{10}$, $P(R|B_2) = \frac{3}{7}$ and $P(R|B_3) = \frac{1}{7}$.
So

$$\begin{aligned} P(R) &= \left(\frac{8}{10} + \frac{3}{7} + \frac{1}{7}\right) \times \frac{1}{3} \\ &= \left(\frac{56}{70} + \frac{30}{70} + \frac{10}{70}\right) \times \frac{1}{3} \\ &= \frac{96}{70} \times \frac{1}{3} \\ &= \frac{48}{35} \times \frac{1}{3} \\ &= \frac{16}{35} \end{aligned}$$

5 marks unseen problem

(c) *Given that a red ball as been picked as described in (b) find the probability that the ball came from bag 2?*

Use Bayes' Theorem:

$$P(B_2|R) = \frac{P(R|B_2)P(B_2)}{P(R}$$

$P(R)$ is given in part (b).
Now, $P(R|B_2)P(B_2) = \frac{3}{7} \times \frac{1}{3} = \frac{1}{7}$
Therefore

$$P(B_2|R) = \frac{\frac{1}{7}}{\frac{16}{35}} = \frac{5}{16}$$

4 marks unseen problem

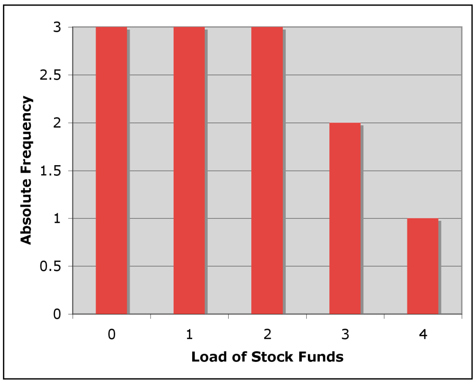
Q5. Consider a sample of size 12 about the load of stock funds.

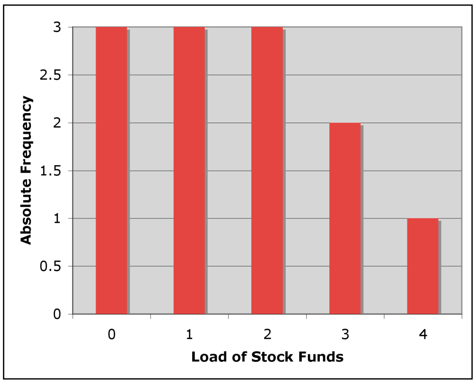
$$0\%,\ 3\%,\ 1\%,\ 3\%,\ 2\%,\ 1\%,\ 0\%,\ 4\%,\ 0\%,\ 2\%,\ 2\%,\ 1\%$$

Calculate the absolute and relative frequency of each load and draw a vertical bar graph for the sample.

value a_j	absolute frequency n_j	relative frequency r_j
0%	3	0.25
1%	3	0.25
2%	3	0.25
3%	2	0.167
4%	1	0.083

Vertical Bar Graph: (Either Absolute or Relative frequency plot is adequate)





3 Marks each for absolute frequency n_j, relative frequency r_j and graph plot

9 Marks Total — unseen problem

Q6. Consider the following sample.

$$0,\ 9,\ 3,\ 2,\ 4,\ 7,\ 3,\ 4,\ 5,\ 4,\ 3,\ 5,\ 5,\ 5,\ 1$$

(a) Calculate the arithmetic mean $\bar{x}$ *and the* sample variance s^2.
There are **15 data samples**
arithmetic mean:

$$\begin{aligned}\bar{x} &= (0+9+3+2+4+7+3+4+5+4+3+5+5+5+1)/15 \\ &= 60/15 \\ &= 4\end{aligned}$$

2 Marks for arithmetic mean

The sample *variance s^2* is defined as

$$s^2 := \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

but also

$$s^2 := \frac{1}{n-1}(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2)$$

which is easier to compute,
So

$$\begin{aligned}s^2 &= \frac{1}{14}((0^2+9^2+3^2+2^2+4^2+7^2+3^2+4^2+5^2+4^2+3^2+5^2+5^2+5^2+1^2) - 15\times 4^2) \\ &= \frac{1}{14}(310-240) \\ &= \frac{70}{14} \\ &= 5\end{aligned}$$

3 Marks for Variance
5 Marks Total — unseen problem

(b) *Calculate the* inter-quartile range IQR *and the* median x_{med} *of the sample.*

Sort Data:

$$x = \{0,\ 1,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 7,\ 9\}$$

Recap:

The **inter-quartile-range** is defined as the difference of the upper quartile and the lower quartile , i.e.

$$IQR := x_{0.75} - x_{0.25}$$

$$x_\alpha = \begin{cases} x_{[n\alpha+1]} & \text{if } n\alpha \text{ is not an integer} \\ \frac{x_{n\alpha}+x_{n\alpha+1}}{2} & \text{if } n\alpha \text{ is an integer} \end{cases}$$

$[n\alpha + 1] =$ the nearest integer to $n\alpha + 1$ which is lower or equal to $n\alpha$.

- the 0.25-quantile — **the lower quartile**,
- the 0.5-quantile — **the median**
- and the 0.75-quantile — **the upper quartile**.

In our case 0.25×15, 0.5×15 and 0.75×15 is not an integer.

So we get

- the 0.25-quantile — $x_4 = 3$
- the median – $x_8 = 4$
- and the 0.75-quantile — $x_{12} = 5$

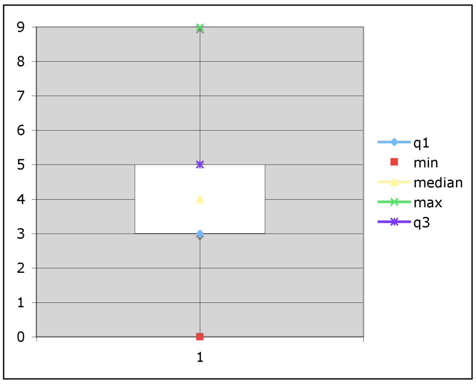
So $IQR = 5 - 3 = 2$ and the median is 4.

3 Marks for IQR, 1 Mark for Median

4 Marks Total — unseen problem

(c) *Draw a* box-plot *for the sample. Are there any* outliers*?*

Box Plot:



3 Marks

Outliers:
We say that a value x_i is an *outlier* if:

$$x_i > x_{0.75} + 1.5 \times IQR := z_u$$

or if

$$x_i < x_{0.25} - 1.5 \times IQR := z_l$$

So in our case
we have an outlier if:

$$x_i > 5 + 1.5 \times 2 = 8$$

or if

$$x_i < 3 - 1.5 \times 2 = 0$$

So we have **ONE** Outlier in data sample 9.
3 Marks
6 Marks Total — unseen problem

Q7. Consider the following sample of returns on stock funds.

$$4.5\%,\ 2.8\%,\ 7.8\%,\ 6.5\%,\ 1.3\%,\ 0.6\%,\ 7.3\%,\ 2.5\%,$$
$$4.7\%,\ 3.2\%,\ 4.9\%,\ 6.9\%,\ 7.2\%,\ 4.6\%,\ 8.7\%$$

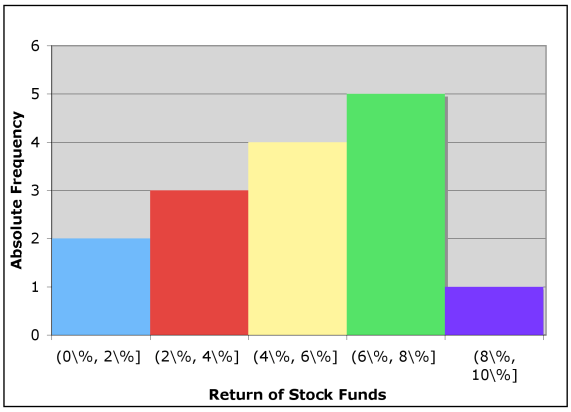
Divide the sample into classes of width 2 *and draw the corresponding* histogram. *Make a statement about the* modality *and the* skewness *of the histogram.*

Note: absolute frequency probably the most likely solution but relative frequency would be acceptable: ONLY ONE of these required for solution

Need a count of frequencies to construct histogram:

class I_j	absolute frequency n_j	relative frequency r_j
(0%, 2%]	2	2/15
(2%, 4%]	3	4/15
(4%, 6%]	4	1/3
(6%, 8%]	5	1/5
(8%, 10%]	1	1/15

This gives the histogram as:



6 Marks for histogram construction

The histogram is clearly skewed *negatively*

6 Marks for correct histogram skew

8 Marks Total — unseen problem

Q8. *Given the following vectors:*

$$\vec{v} = (2, 4), \vec{w} = (1, 6)$$

(a) What are the norms *of $\vec{v}$ and $\vec{w}$?*

Norm of $\|v\| = \sqrt{2^2 + 4^2} = \sqrt{4 + 16} = \sqrt{20} = 4.47$
Norm of $\|w\| = \sqrt{1^2 + 6^2} = \sqrt{1 + 36} = \sqrt{37} = 6.08$

2 Marks — unseen problem

(b) What is the scalar product *$\vec{v}.\vec{w}$?*

$v.w = 2 \times 1 + 4 \times 6 = 2 + 24 = 26.$
2 Marks — unseen problem

(c) What is the angle θ between $\vec{v}$ and $\vec{w}$?

$$v.w = \|v\| \|w| \cos(\theta)$$

so

$$\cos(\theta) = \frac{v.w}{\|v\| \|w|}$$

from (a) and (b)

$$\cos(\theta) = \frac{26}{4.47 * 6.09} = 0.957$$

So $\theta = \cos^{-1}(0.958) = 16.93°$

3 Marks — unseen problem

(d) What is the vector cross product *$\vec{v} \times \vec{w}$?*

Let $n = 2$. We define the vector product of $v, w \in \mathbb{R}^2$ as a map $\times$: $\mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ with

$$v \times w = v_1 w_2 - v_2 w_1$$

So we get $v \times w = 2 * 6 - 4 * 1 = 12 - 4 = 8$

4 Marks — unseen problem

Q9. Calculate the determinant of the matrix

$$B = \begin{pmatrix} -1 & 4 & 2 \\ -2 & 5 & 3 \\ -3 & 0 & -7 \end{pmatrix}$$

$$\begin{aligned}
\det B &= \begin{vmatrix} -1 & 4 & 2 \\ -2 & 5 & 3 \\ -3 & 0 & -7 \end{vmatrix} \\
&= -1 \times \begin{vmatrix} 5 & 3 \\ -0 & -7 \end{vmatrix} - 4 \times \begin{vmatrix} -2 & 3 \\ -3 & -7 \end{vmatrix} + 2 \times \begin{vmatrix} -2 & 5 \\ -3 & 0 \end{vmatrix} \\
&= -1 \times ((5).(-7) - 3.0) - 4 \times ((-2).(-7) - (-3).3) + 2 \times ((-2).0 - (-3).5) \\
&= -1 \times -35 - 4 \times 23 + 2 \times 15 \\
&= 35 - 92 + 30 \\
&= -27
\end{aligned}$$

7 Marks — unseen problem

END OF EXAMINATION