CARDIFF UNIVERSITY
EXAMINATION PAPER

Academic Year: 2007/2008

Examination Period: Spring

Examination Paper Number: CM0167Solutions

Examination Paper Title: Mathematics for Computer Science

SOLUTIONS

Duration: 2 hours

Do not turn this page over until instructed to do so by the Senior Invigilator.

Structure of Examination Paper:

There are 21 pages.
There are 9 questions in total.
There are no appendices.
The mark obtainable for a question or part of a question is shown in brackets alongside the question.

Students to be provided with:

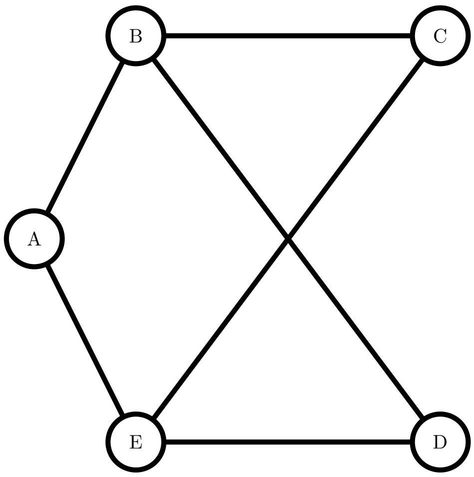
The following items of stationery are to be provided:
ONE answer book.

Instructions to Students:

Answer all questions.
The use calculators **without** programmable memory is permitted.
The use of translation dictionaries between English or Welsh and a foreign language bearing an appropriate departmental stamp is permitted in this examination.

Q1. Given the following vertex set, $V = \{A, B, C, D, E\}$*, and edge set,* $E = \{AB, AE, BC, BD, CE, DE\}$ *for a simple graph,* $G = (V, E)$*:*

(a) Draw *the graph,* G.



3 Marks — *Clearly Many Drawing Variations possible must have same topology of course*

(b) What is the order *and* size *of the graph,* G

Order = number of vertices = 5 [1]

Size = number of edges = 6 [1]

2 Marks

(c) What is the adjacency matrix *for the graph,* G.

$$
\begin{array}{lc}
 & \overbrace{\begin{array}{ccccc} A & B & C & D & E \end{array}}^{\text{Cols } 1\ldots n(=5)} \\
\begin{array}{l} \text{(Row 1)} \quad A \\ \text{(Row 3)} \quad B \\ \text{(Row 3)} \quad C \\ \text{(Row 4)} \quad D \\ \text{(Row 5)} \quad E \end{array} &
\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}
\end{array}
$$

3 Marks

8 Marks Question Total — Unseen Problem

Q2. Using the HuffmanCoding Algorithm *code the following sequence of characters:*

`ABBAACCAADDA`

Letter count for above is

A	6
B	2
C	2
D	2

Applying Huffman Coding Algorithm:

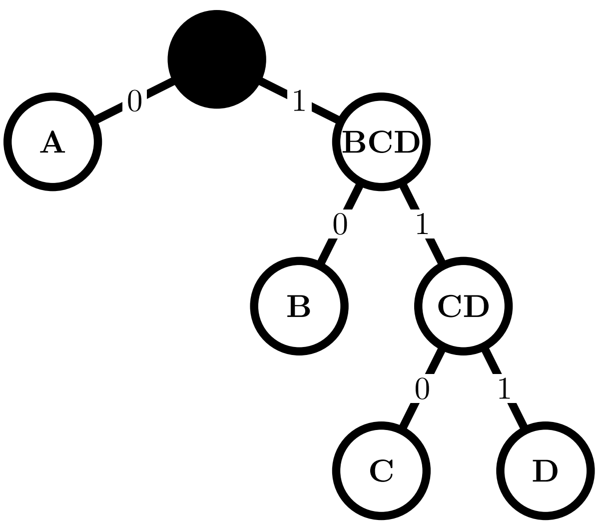
So first we merge C and D to get:

A	6
B	2
CD	4

Then Merge B and CD to get:

A	6
BCD	6

So tree is:



so the codes for the letters are:

A	0
B	10
C	110
D	111

So sequence is:

$$\overbrace{0}^{\mathtt{A}}\;\overbrace{10}^{\mathtt{B}}\;\overbrace{10}^{\mathtt{B}}\;\overbrace{0}^{\mathtt{A}}\;\overbrace{0}^{\mathtt{A}}\;\overbrace{110}^{\mathtt{C}}\;\overbrace{110}^{\mathtt{C}}\;\overbrace{0}^{\mathtt{A}}\;\overbrace{0}^{\mathtt{A}}\;\overbrace{111}^{\mathtt{D}}\;\overbrace{111}^{\mathtt{D}}\;\overbrace{0}^{\mathtt{A}}$$

3 Marks For Sort
3 Marks For Tree
2 Marks For Coding Sequence
8 Marks Question Total — Unseen Problem

Q3. : Consider the following table of distances between the cities A, B, C, D and E:

	A	B	C	D	E
A	–	22	20	21	34
B	22	–	47	51	38
C	20	47	–	18	33
D	21	51	18	–	71
E	34	38	33	71	–

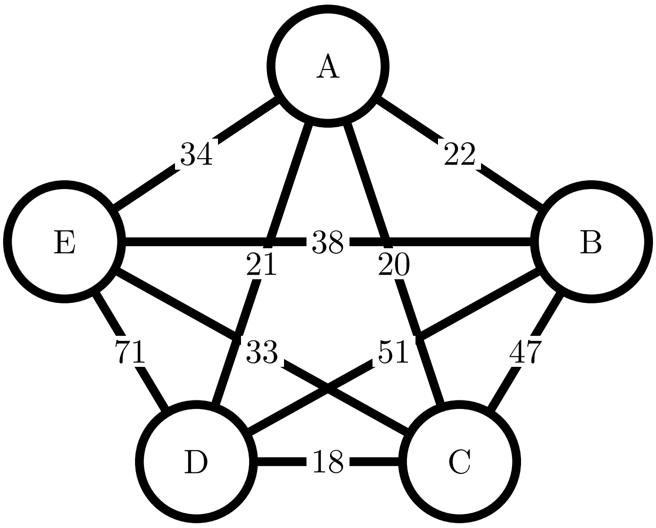
(a) Draw a graph to represent the information in the table above. [4]

(b) Find an upper bound *for the solution to the travelling salesman problem for the six cities above using the heuristic* nearest neighbour *algorithm.* [7]

(c) Find a lower bound *for the solution to the travelling salesman problem by* removing *city A.* [7]

(a) Graph of table

The Graph representation for the above table is:

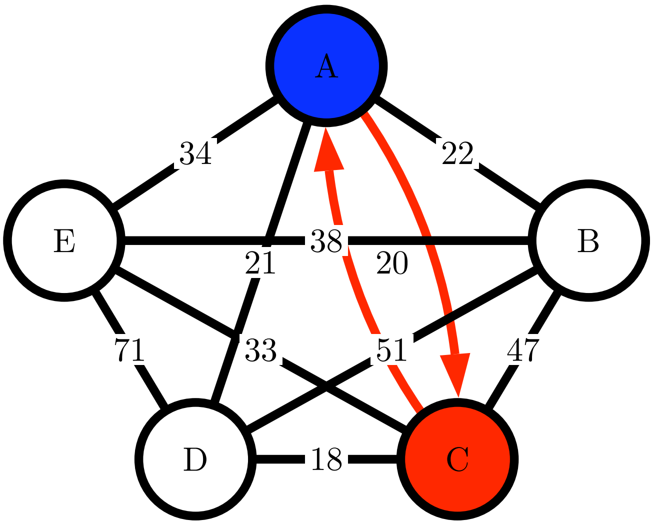


4 Marks

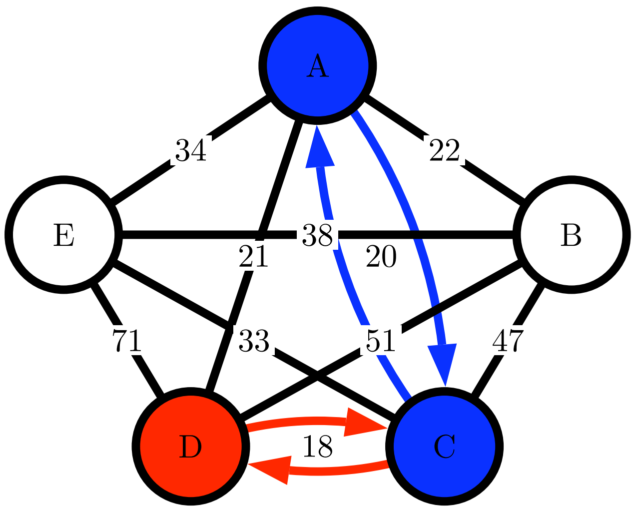
(b) Upper Bound Solution

To find the *upper bound* use the *heuristic (nearest neighbour) algorithm*:

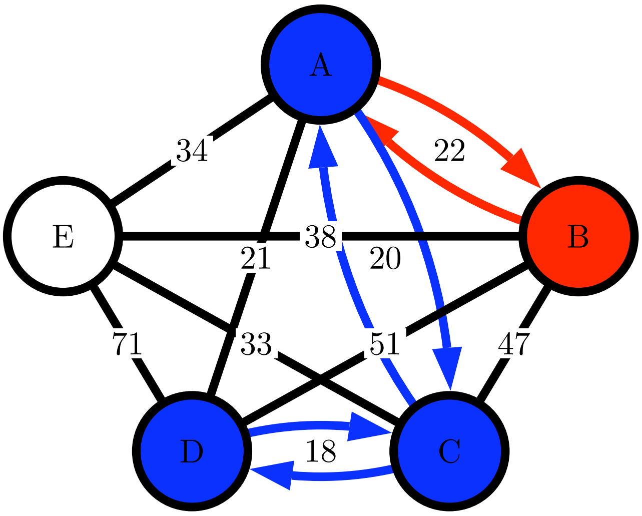
(a) Choose a vertex, say A (Note you get a different but valid solution if you start from another vertex, Draw A Lowest weight is AC so draw this as a cycle of clockwise directional arcs and draw C:



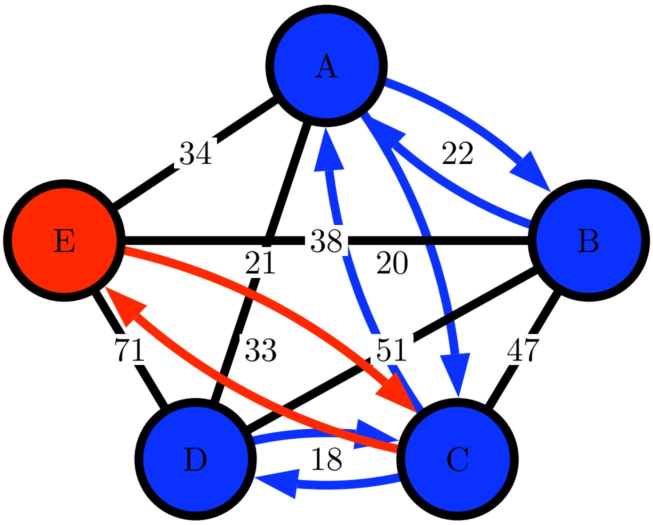
(b) Vertices, A and C drawn. Lowest weight is DC so draw this a clockwise cycle and draw D:



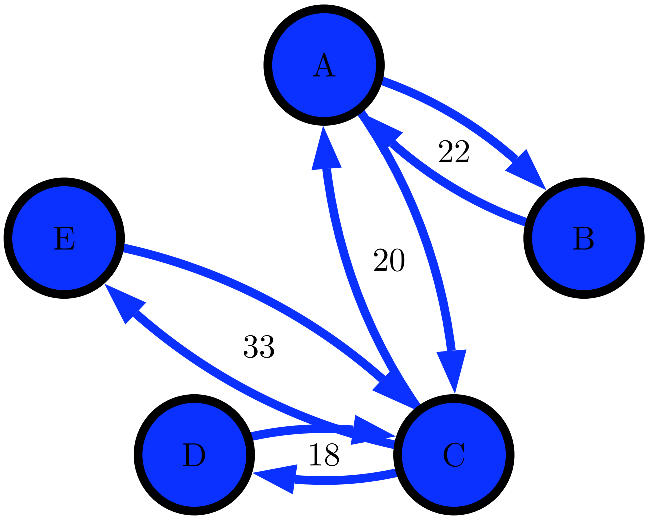
(c) Vertices, A, C and D drawn. Lowest weight is DA **but** D **and** A **already drawn**, Lowest weight to *undrawn vertex* is AB so draw this a clockwise cycle and draw B:



(d) Vertices, A, C, D and B drawn. Only Vertex E undrawn. Lowest weight to E is CE so draw this a clockwise cycle and draw E:



So *Hamiltonian Cycle* is given by:



The **Upper BOUND** for the TSP of this problem is the weight of this cycle which is:

$$2 \times (22 + 20 + 18 + 33) = 186$$

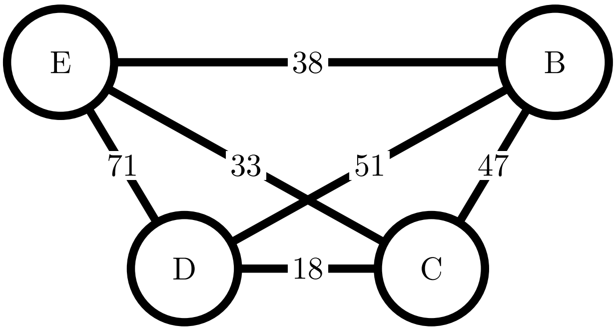
1 mark for each step plus 1 marks for final cycle graph plus 2 marks for Upper bound calculation

7 Marks total — unseen problem

(c) Lower Bound Solution

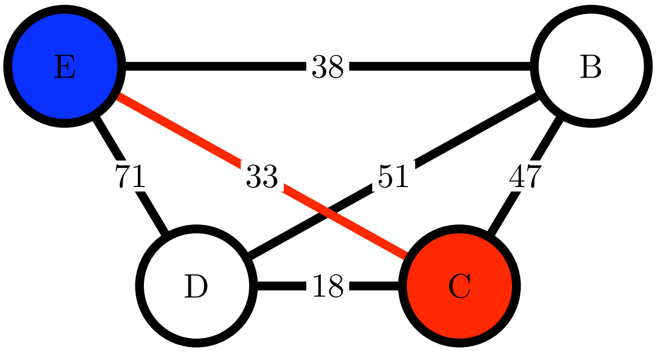
To find the *lower bound* use the *lower bound algorithm*:

(a) Choose a vertex, say A (Note you get a different but valid solution if you start from another vertex), Remove A from graph.



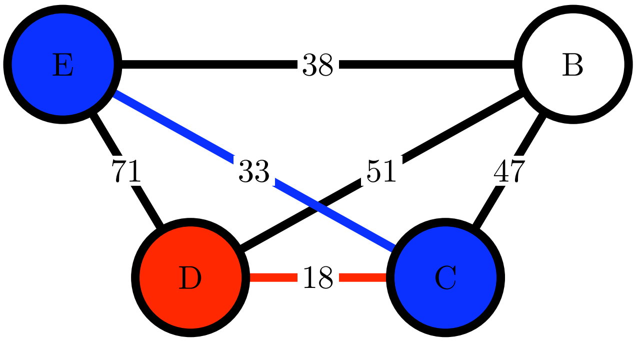
(b) Find Minimum Spanning Tree via *Prim's Algorithm* :

i. Choose vertex E, draw$E$. Lowest weighted edge is EC so draw this edge and vertex C



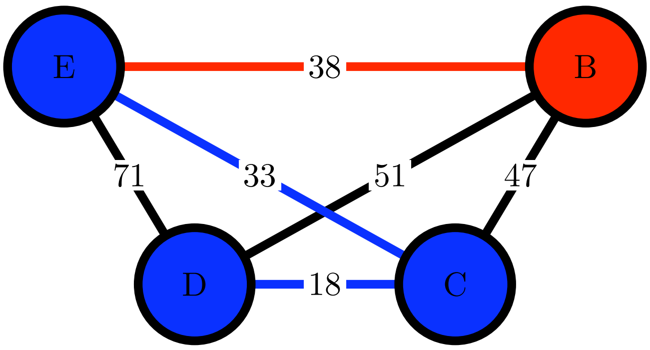
[1]

ii. Vertices E and C drawn. Lowest weighted edge is CD so draw this edge and vertex D



[1]

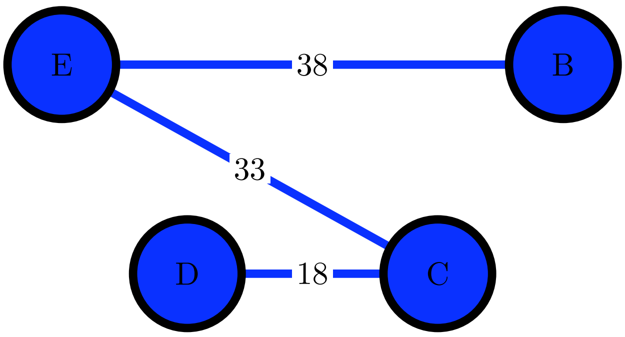
iii. Vertices E, C and D drawn. Lowest weighted edge is EB so draw this edge and vertex B



[1]

So Minimum Spanning Tree is:

(c) Vertices E and C drawn. Lowest weighted edge is CD so draw this edge and vertex D



[1]

So the weight of this tree,w is: 18 + 38 + 33 = 89. [1]

Now we need to find the two lowest weight connecting A.

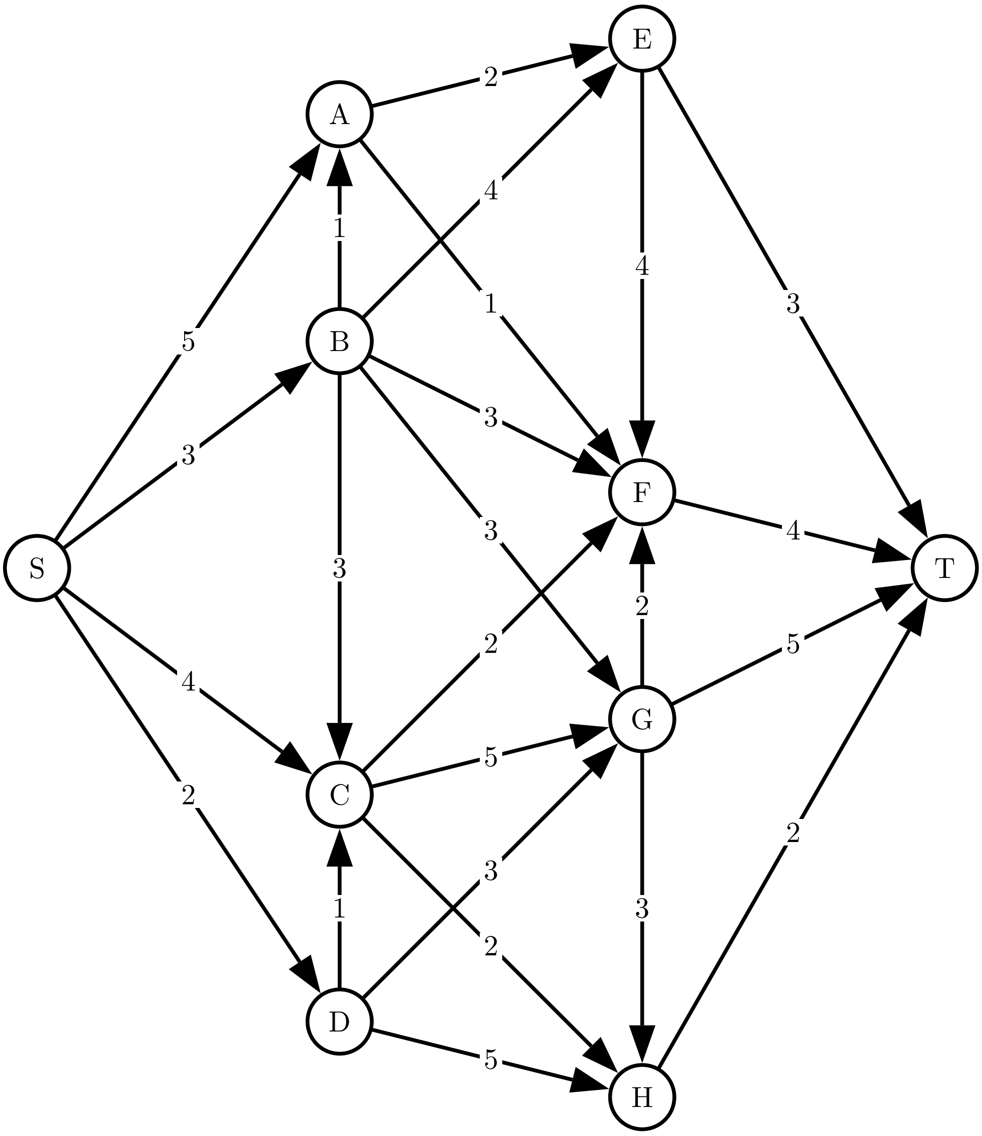
These are AC, $w_1 = 20$ and AD, $w_2 = 21$.

So the **Lower Bound for this TSP** problem is: $w + w_1 + w_2 = 89 + 20 + 21 = 130$ [1]

1 mark for each step, 2 Marks for Lower Bound calculation — 1 mark for tree weight plus 1 mark for lower bound calculation

7 Marks total — unseen problem

Q4. Find the shortest path from S to T in the digraph below using Dijkstra's algorithm. Show your working with tables.



Step	Vertex marked	Current potential	S	A	B	C	D	E	F	G	H	T	Unchosen vertices
			Distance to Vertex										
1	S	0	0	5	3	4	2	–	–	–	–	–	A,B,C,D,E,F,G,H,T
2	D	2	0	5	3	3	2	–	–	5	7	–	A,B,C,E,F,G,H,T
3	B	3	0	4	3	3	2	7	6	5	7	–	A,C,E,F,G,H,T
4	C	3	0	4	3	3	2	7	5	5	5	–	A,E,F,G,H,T
5	A	4	0	4	3	3	2	6	5	5	5	–	E,F,G,H,T
6	F	5	0	4	3	3	2	6	5	5	5	9	E,G,H,T
7	G	5	0	4	3	3	2	6	5	5	5	9	E,H,T
8	H	5	0	4	3	3	2	6	5	5	5	7	E,T
8	E	6	0	4	3	3	2	6	5	5	5	7	T

This is explained as follows (**= choose**, **= chosen**, **= dont overwrite**):

Step 1 — Only Valid Paths from S are to vertices A, B C and D. Add weights in table choose lowest with is D

Step 2 — Update Current Potential (2). D can link to G and H potential to C is lower than current so can update.B is lowest choose this.

Step 3 — Update Current Potential (3). B can link to E and F which are new and G which is higher than current potential. B can also link to A with a lower potential of 4 so change this. C is lowest choose this.

Step 4 — Update Current Potential (3). C can link to$E$ and$F$ but with a higher than current potential. C can also link to H with a lower potential of 5 so change this. A is lowest choose this.

Step 5 — Update Current Potential (4). A can link to F with same potential and G which is higher than current potential. A can link to E with lower potential so change. F is lowest choose this.

Step 6 — Update Current Potential (5). F can link to T so add this. G is lowest choose this.

Step 7 — Update Current Potential (5). Cost of getting to any of G's links is higher than any current potential. H is lowest choose this.

Step 8 — Update Current Potential (5). Cost of getting to T from H is less, 7, so change this, E is lowest choose this.

Step 9 — Update Current Potential (6). Cost of getting to T or F from H is higher, Only T left.

Shortest path following back from T is: $SDCHT$.

1 Mark per step + 1mark for shortest path

10 Marks TOTAL — unseen problem

Q5. A scout group contains 2 adult scoutmasters and 10 boy scouts. They are invited to send four members to a scout convention

(a) Evaluate the number of ways *that the group may be selected so that it includes both scoutmasters?*

If the group contains **both** scoutmasters then the number of ways of choosing 2 from 10 is ${}^{10}C_2$.

$$^{10}C_2 = \frac{10.9}{2.1} = 45$$

Therefore there are **45 ways** to choose **both scoutmasters** in the group.

2 Marks

(b) Evaluate the number of ways *that the group may be selected so that it includes only one scoutmaster?*

The number of ways to choose **one** scoutmasters $= 2$.

The number of ways of choosing the remaining 3 from 10 is ${}^{10}C_3$.

$$^{10}C_3 = \frac{10.9.8}{3.2.1} = 120$$

Therefore the number of ways to choose this group of 4 is $2.120 = 240$

Therefore there are **240 ways** to choose **only one scoutmaster** in the group of 4.

3 Marks

(c) Evaluate the number of ways that the group may be selected so that it includes neither scoutmaster?

Simply need to choose any 4 from the 10 boy scouts.

Number of ways to choose 4 from 10 is ${}^{10}C_4$.

$$^{10}C_4 = \frac{10.9.8.7}{4.3.2.1} = 210$$

Therefore there are **210 ways** to choose **no scoutmaster** in the group of 4.

3 Marks

(d) *On a different occasion, the boy scouts decide to play five-a-side football. Each team is chosen by a random selection of all the 10 boy scouts. Given that there are two brothers in the group what is probability that the two brothers will be picked in the same team?*

Number of ways to choose 5 from 10 is ${}^{10}C_5$.

$$ {}^{10}C_5 = \frac{10.9.8.7.6}{5.4.3.2.1} = 252 $$

As there are only 10 boys in all. When one team of **five** is chosen the other team of five is chosen *automatically*.

HOWEVER the teams maybe ***interchangeable***: That is the team chosen as ABCDE versus FGHIJ can also be selected as FGHIJ versus ABCDE!

Therefore we must ***half*** the number of choices from 5 from 10 for the team selection: $\frac{252}{2} = 126$

Therefore the teams can be formed in 126 ways

Given the two brothers are in the same team then the number of ways the remaining 8 can be chosen is ${}^{8}C_3$

$$ {}^{8}C_3 = \frac{8.7.6}{3.2.1} = 56 $$

SO the ***probability that the two brothers are in the same team***, P(E) is

$$ P(E) = \frac{\text{Number of ways two brothers in the team}}{\text{Total number of ways teams can be selected (5 from 10)}} $$

so

$$ P(E) = \frac{56}{126} = \frac{4}{9} $$

5 Marks

13 Marks for TOTAL Question

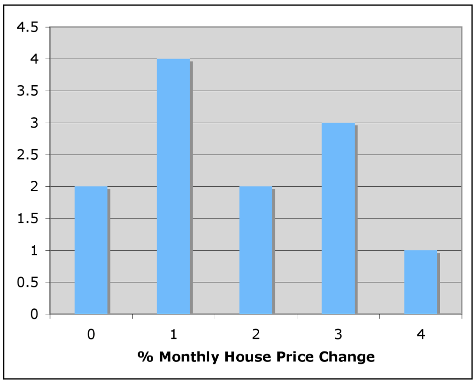
Q6. Consider a sample of size 12 about the monthly change in house prices.

$$0\%,\ 1\%,\ 3\%,\ 3\%,\ 2\%,\ 1\%,\ 0\%,\ 1\%,\ 3\%,\ 4\%,\ 2\%,\ 1\%$$

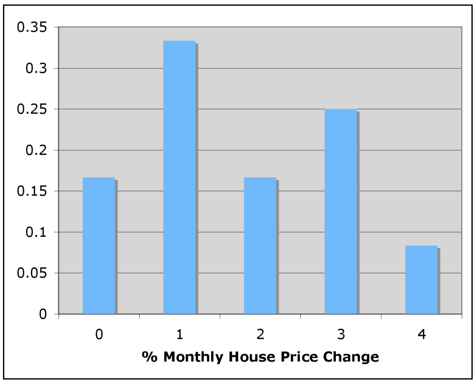
Calculate the absolute *and* relative frequency *of each monthly change and draw a vertical bar graph for the sample.*

value a_j	absolute frequency n_j	relative frequency r_j
0%	2	0.167
1%	4	0.333
2%	2	0.167
3%	3	0.25
4%	1	0.083

Vertical Bar Graph: (Either Absolute or Relative frequency plot is adequate)



Absolute Frequency



Relative Frequency

3 Marks each for absolute frequency n_j, relative frequency r_j and graph plot

9 Marks for TOTAL Question — unseen problem

Q7. *Consider the following sample.*

$$0,\ 3,\ 5,\ 2,\ 9,\ 7,\ 3,\ 5,\ 6,\ 4,\ 3,\ 2,\ 4,\ 5,\ 2$$

(a) *Calculate the* arithmetic mean $\bar{x}$ *and the* sample variance s^2.

There are **15 data samples**

arithmetic mean:

$$\begin{aligned}
\bar{x} &= (0+3+5+2+9+7+3+5+6+4+3+2+4+5+2)/15 \\
&= 60/15 \\
&= 4
\end{aligned}$$

2 Marks for arithmetic mean

The sample *variance s^2* is defined as

$$s^2 := \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

but also

$$s^2 := \frac{1}{n-1}(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2)$$

which is easier to compute,

So

$$\begin{aligned}
s^2 &= \frac{1}{14}((0^2+3^2+5^2+2^2+9^2+7^2+3^2+5^2+6^2+4^2+3^2+2^2+4^2+5^2+2^2) \\
&\quad -15 \times 4^2) \\
&= \frac{1}{14}(314-240) \\
&= \frac{74}{14} \\
&= 5.29
\end{aligned}$$

3 Marks for Variance

5 Marks Total — unseen problem

(b) *Calculate the* inter-quartile range IQR *and the* median x_{med} *of the sample.*

Sort Data:

$$x = \{0,\ 2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 5,\ 5,\ 5,\ 6,\ 7,\ 9\}$$

Recap:

The **inter-quartile-range** is defined as the difference of the upper quartile and the lower quartile , i.e.

$$IQR := x_{0.75} - x_{0.25}$$

$$x_\alpha = \begin{cases} x_{[n\alpha+1]} & \text{if } n\alpha \text{ is not an integer} \\ \frac{x_{n\alpha}+x_{n\alpha+1}}{2} & \text{if } n\alpha \text{ is an integer} \end{cases}$$

$[n\alpha + 1] =$ the nearest integer to $n\alpha + 1$ which is lower or equal to $n\alpha$.

- the 0.25-quantile — **the lower quartile**,
- the 0.5-quantile — **the median**
- and the 0.75-quantile — **the upper quartile**.

In our case 0.25×15, 0.5×15 and 0.75×15 is not an integer.

So we get

- the 0.25-quantile — $x_4 = 2$
- the median – $x_8 = 4$
- and the 0.75-quantile — $x_{12} = 5$

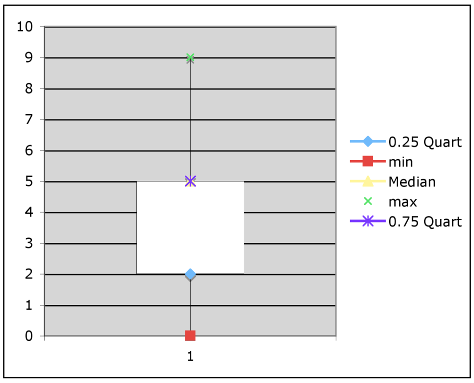
So $IQR = 5 - 2 = 3$ and the median is 4.

3 Marks for IQR, 1 Mark for Median

4 Marks Total — unseen problem

(c) *Draw a* box-plot *for the sample. Are there any* outliers*?*

Box Plot:



3 Marks

Outliers:
We say that a value x_i is an *outlier* if:

$$x_i > x_{0.75} + 1.5 \times IQR := z_u$$

or if

$$x_i < x_{0.25} - 1.5 \times IQR := z_l$$

So in our case
we have an outlier if:

$$x_i > 5 + 1.5 \times 3 = 9.5$$

or if

$$x_i < 2 - 1.5 \times 3 = -2.5$$

So we have **NO** OutlierS in THIS data sample.
3 Marks
6 Marks Total — unseen problem

15 Marks for TOTAL Question

Q8. *Given the following vectors:*

$$\mathbf{v} = (3, 5), \mathbf{w} = (1, -4)$$

(a) What are the norms *of $\vec{v}$ and $\vec{w}$?*

Norm of $\|v\| = \sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \sqrt{34} = 5.83$
Norm of $\|w\| = \sqrt{1^2 + -4^2} = \sqrt{1 + 16} = \sqrt{17} = 4.12$

2 Marks — unseen problem

(b) What is the scalar product *$\vec{v}.\vec{w}$?*

$v.w = 3 \times 1 + 5 \times -4 = 3 - 20 = -17.$
2 Marks — unseen problem

(c) What is the angle θ between $\vec{v}$ and $\vec{w}$?

$$v.w = \|v\| \|w| \cos(\theta)$$

so

$$\cos(\theta) = \frac{v.w}{\|v\| \|w|}$$

from (a) and (b)

$$\cos(\theta) = \frac{-17}{5.83 * 4.12} = -0.707$$

So $\theta = \cos^{-1}(0.958) = 134.99°$

3 Marks — unseen problem

(d) What is the vector cross product *$\vec{v} \times \vec{w}$?*

Let $n = 2$. We define the vector product of $v, w \in \mathbb{R}^2$ as a map $\times$: $\mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ with

$$v \times w = v_1 w_2 - v_2 w_1$$

So we get $v \times w = 3 * -4 - 5 * 1 = -12 - 5 = -17$

4 Marks — unseen problem

(e) What is the area of the parallelogram spanned by $\mathbf{v}$ *and* $\mathbf{w}$*?*
Area of parallelogram is $|\vec{v} \times \vec{w}| = 17$
2 Marks — unseen problem

13 Marks for TOTAL Question

Q9. Calculate the determinant of the matrix

$$A = \begin{pmatrix} 4 & 2 & 4 \\ -1 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

Can do determinant decomposition by any row (or column). As there is a zero in third row. We can exploit this to work out determinant more easily:

$$\begin{aligned}
\det A &= \begin{vmatrix} 4 & 2 & 4 \\ -1 & 1 & 3 \\ 2 & 0 & 1 \end{vmatrix} \\
&= 2 \times \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} - \mathbf{0} \times \begin{vmatrix} 4 & 4 \\ -1 & 3 \end{vmatrix} + 1 \times \begin{vmatrix} 4 & 2 \\ -1 & 1 \end{vmatrix} \\
&= 2 \times (2.3 - 4.1) + 1 \times (4.1 - (-1).2) \\
&= 2 \times 2 + 1 \times 6 \\
&= 4 + 6 \\
&= 10
\end{aligned}$$

7 Marks — unseen problem

7 Marks for TOTAL Question