CARDIFF UNIVERSITY
EXAMINATION PAPER

Academic Year:	2008/2009
Examination Period:	Spring
Examination Paper Number:	CM0167Solutions
Examination Paper Title:	Mathematics for Computer Science

SOLUTIONS

Duration:	2 hours

Do not turn this page over until instructed to do so by the Senior Invigilator.

Structure of Examination Paper:

There are 22 pages.
There are 9 questions in total.
There are no appendices.
The mark obtainable for a question or part of a question is shown in brackets alongside the question.

Students to be provided with:

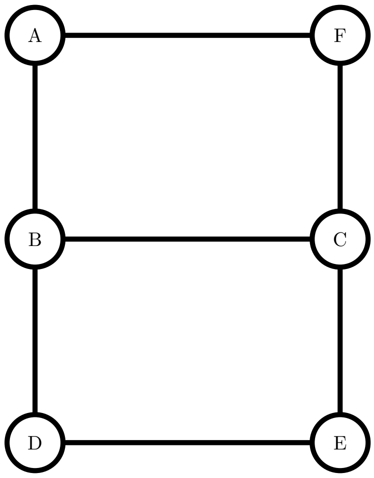
The following items of stationery are to be provided:
ONE answer book.

Instructions to Students:

Answer all questions.
The use calculators **without** programmable memory is permitted.
The use of translation dictionaries between English or Welsh and a foreign language bearing an appropriate departmental stamp is permitted in this examination.

Q1. Given the following vertex set, $V = \{A, B, C, D, E, F\}$, and edge set, $E = \{AB, AF, BC, BD, CE, CF, DE\}$ for a simple graph, $G = (V, E)$:

(a) Draw *the graph, G.*



3 Marks — *Clearly Many Drawing Variations possible must have same topology of course*

(b) What is the order *and* size *of the graph, G*

Order = number of vertices = 6 [1]

Size = number of edges = 7 [1]

2 Marks

(c) What is the adjacency matrix *for the graph, G.*

$$\begin{array}{rc} & \overbrace{\begin{array}{cccccc} A & B & C & D & E & F \end{array}}^{\text{Cols } 1\ldots n(=6)} \\ \begin{array}{rc} \text{(Row 1)} & A \\ \text{(Row 2)} & B \\ \text{(Row 3)} & C \\ \text{(Row 4)} & D \\ \text{(Row 5)} & E \\ \text{(Row 6)} & F \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

3 Marks

8 Marks Question Total — Unseen Problem

Q2. Using the HuffmanCoding Algorithm *code the following sequence of characters:*

`ABBBBCCAADDA`

Letter count for above is

A	4
B	4
C	2
D	2

Applying Huffman Coding Algorithm:

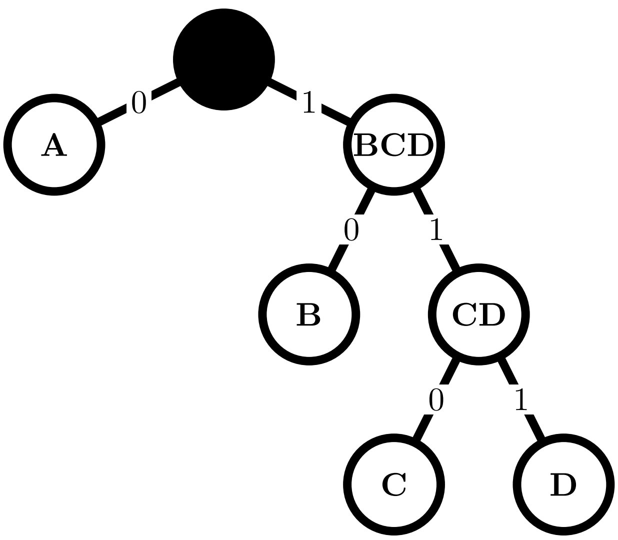
So first we merge C and D to get:

A	4
B	4
CD	4

Then Merge B and CD to get:

A	4
BCD	8

So tree is:



so the codes for the letters are:

A	0
B	10
C	110
D	111

So sequence is:

$$\overbrace{0}^{\texttt{A}}\ \overbrace{10}^{\texttt{B}}\ \overbrace{10}^{\texttt{B}}\ \overbrace{10}^{\texttt{B}}\ \overbrace{10}^{\texttt{B}}\ \overbrace{110}^{\texttt{C}}\ \overbrace{110}^{\texttt{C}}\ \overbrace{0}^{\texttt{A}}\ \overbrace{0}^{\texttt{A}}\ \overbrace{111}^{\texttt{D}}\ \overbrace{111}^{\texttt{D}}\ \overbrace{0}^{\texttt{A}}$$

3 Marks For Sort
3 Marks For Tree
2 Marks For Coding Sequence
8 Marks Question Total — Unseen Problem

Q3. Consider the following table of distances between the cities A, B, C, D and E:

	A	*B*	*C*	*D*	*E*
A	–	*24*	*16*	*29*	*23*
B	*24*	–	*37*	*41*	*58*
C	*16*	*37*	–	*14*	*23*
D	*29*	*41*	*14*	–	*31*
E	*23*	*58*	*23*	*31*	–

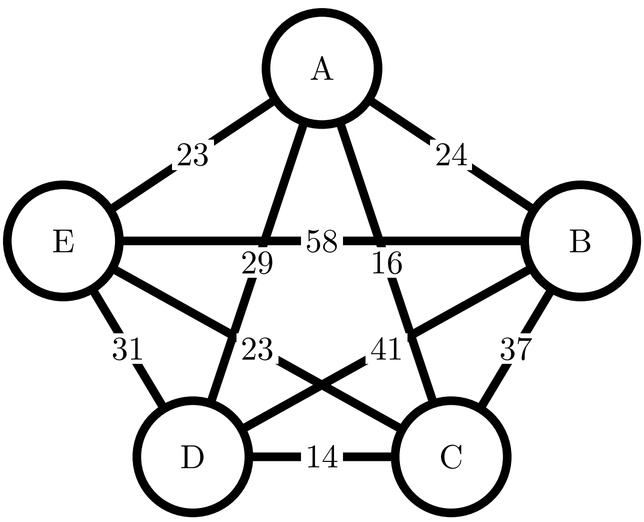
(a) Draw a graph to represent the information in the table above. *[4]*

(b) Find an upper bound *for the solution to the travelling salesman problem for the six cities above using the heuristic* nearest neighbour *algorithm.* *[7]*

(c) Find a lower bound *for the solution to the travelling salesman problem by* removing *city A.* *[7]*

(a) Graph of table

The Graph representation for the above table is:

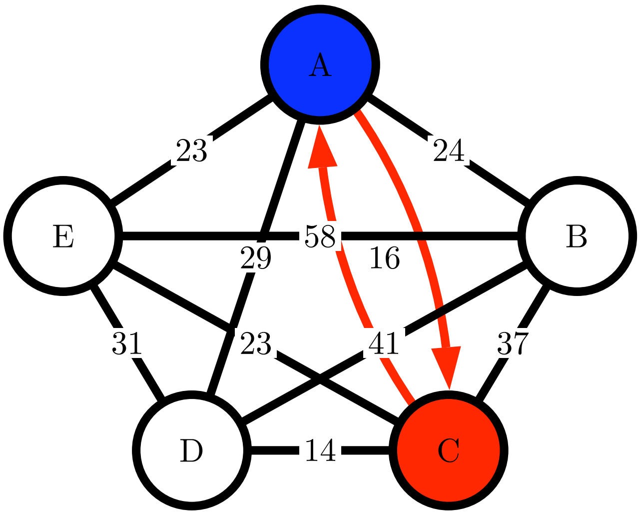


4 Marks

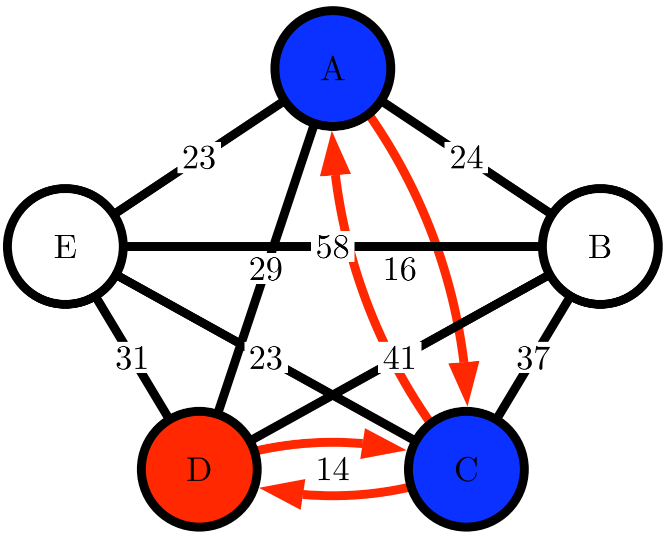
(b) Upper Bound Solution

To find the *upper bound* use the *heuristic (nearest neighbour) algorithm*:

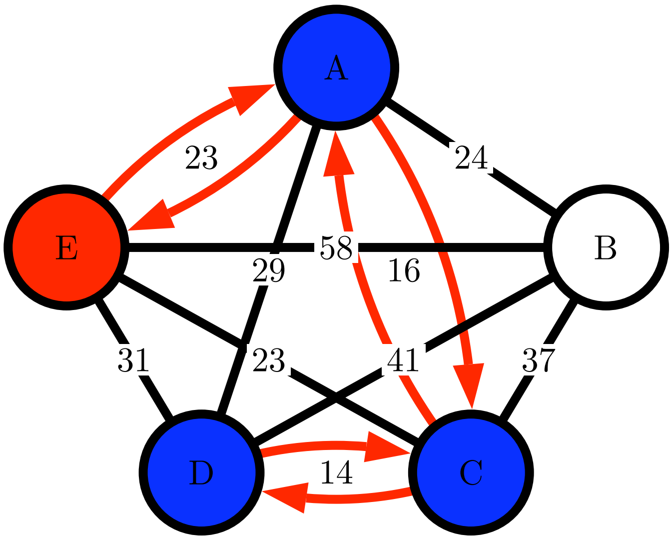
(a) Choose a vertex, say A (Note you get a different but valid solution if you start from another vertex, Draw A Lowest weight is AC so draw this as a cycle of clockwise directional arcs and draw C:



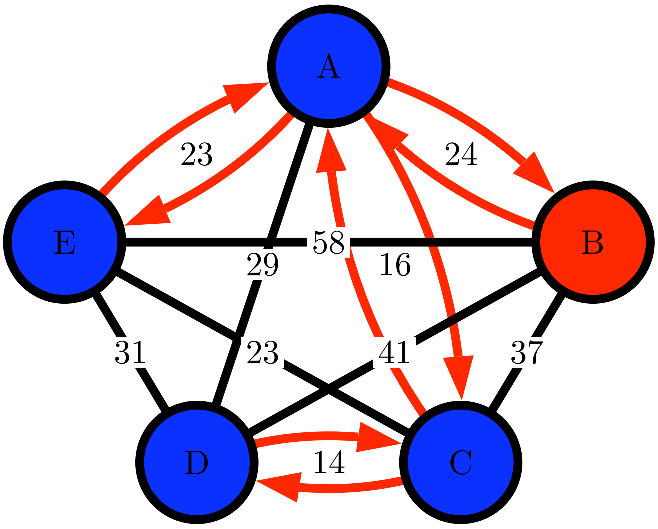
(b) Vertices, A and C drawn. Lowest weight is DC so draw this a clockwise cycle and draw D:



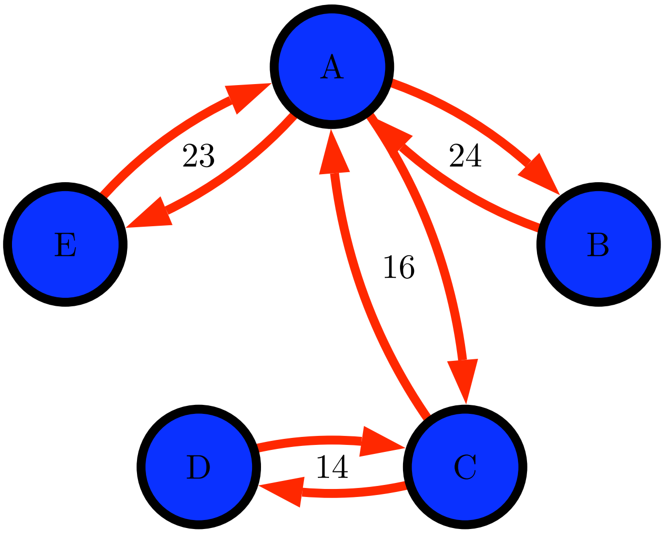
(c) Vertices, A, C and D drawn. Lowest weight is AE so draw this a clockwise cycle and draw E: (CE is equal so could be drawn instead)



(d) Vertices, A, C, D and E drawn. Only Vertex B undrawn. Lowest weight to B is AB so draw this a clockwise cycle and draw B:



So *Hamiltonian Cycle* is given by:



The **Upper BOUND** for the TSP of this problem is the weight of this cycle which is:

$$2 \times (16 + 14 + 23 + 24) = 154$$

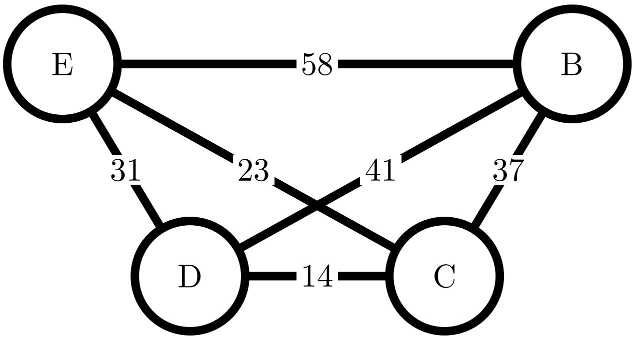
1 mark for each step plus 1 marks for final cycle graph plus 2 marks for Upper bound calculation

7 Marks total — unseen problem

(c) Lower Bound Solution

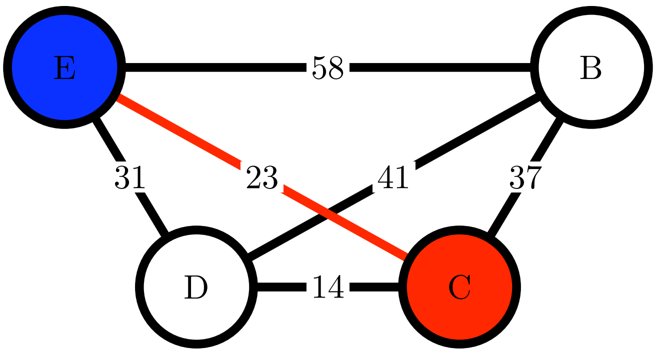
To find the *lower bound* use the *lower bound algorithm*:

(a) Choose a vertex, say A (Note you get a different but valid solution if you start from another vertex), Remove A from graph.



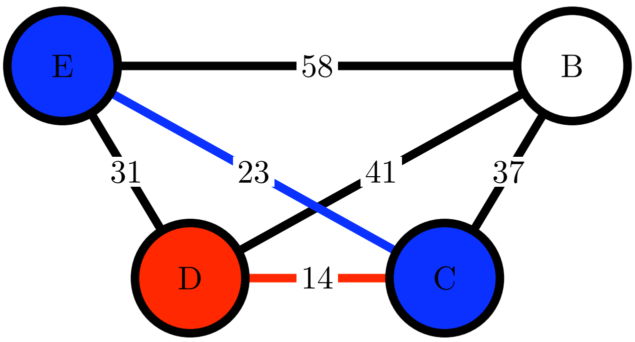
(b) Find Minimum Spanning Tree via *Prim's Algorithm* :

i. Choose vertex E, draw$E$. Lowest weighted edge is EC so draw this edge and vertex C



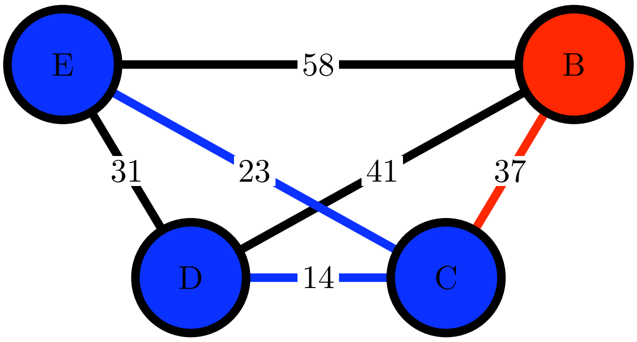
[1]

ii. Vertices E and C drawn. Lowest weighted edge is CD so draw this edge and vertex D



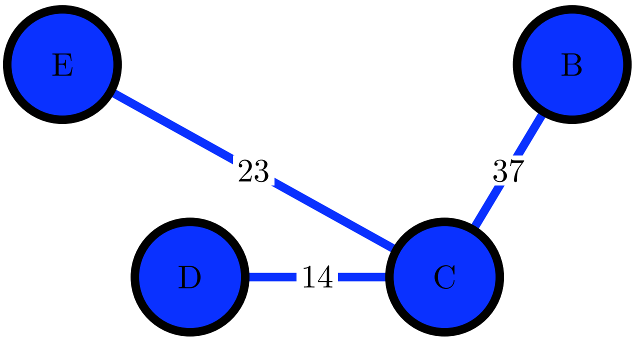
[1]

iii. Vertices E, C and D drawn. Lowest weighted edge is CB so draw this edge and vertex B



[1]

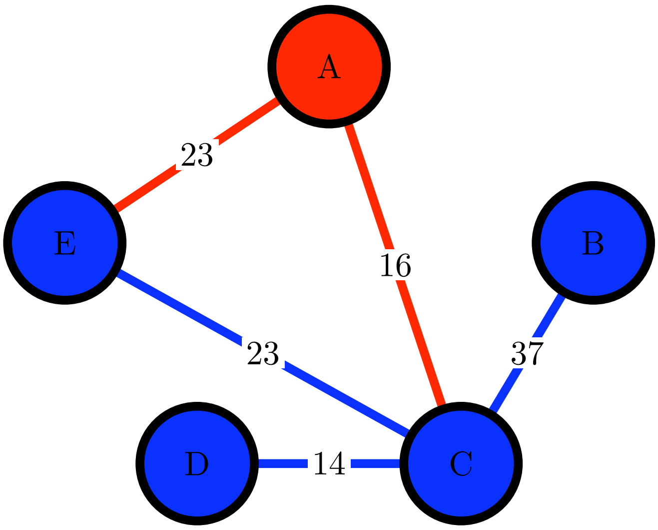
So Minimum Spanning Tree is:



[1]

So the weight of this tree,w is: 23 + 14 + 37 = 74. [1]

Now we need to find the two lowest weight connecting A.
These are AC, $w_1 = 16$ and AE, $w_2 = 23$.

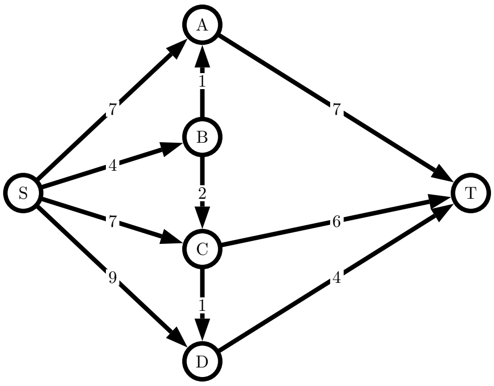


So the **Lower Bound for this TSP** problem is: $w + w_1 + w_2 = 74 + 16 + 23 = 113$ [1]

1 mark for each step, 2 Marks for Lower Bound calculation — 1 mark for tree weight plus 1 mark for lower bound calculation

7 Marks total — unseen problem

Q4. Find the shortest path from S to T in the digraph below using Dijkstra's algorithm. Show your working with tables.



Step	Vertex marked	Current potential	Distance to Vertex S	A	B	C	D	T	Unchosen vertices
1	S	0	0	7	**4**	7	9	–	A,B,C,D,T
2	B	4	0	**5**	4	6	9	–	A,C,D,T
3	A	5	0	**5**	**4**	**6**	9	12	C,D,T
4	C	6	0	**5**	**4**	**6**	**7**	12	D,T
5	D	7	0	**5**	**4**	**6**	**7**	11	T

This is explained as follows (**= choose**, **= chosen**, **= dont overwrite**):

Step 1 — Only Valid Paths from S are to vertices A, B C and D. Add weights in table choose lowest with is B

Step 2 — Update Current Potential (4). B does not link to any new nodes. However, B links to A and to C lower than current so update.A is lowest choose this.

Step 3 — Update Current Potential (5). A can link to T so add this. C is lowest choose this.

Step 4 — Update Current Potential 6). C can link to$T$ but with same potential (12) as A. C can also link to D with a lower potential of 7 so change this. D is lowest choose this.

Step 5 — Update Current Potential (7). D can link to T with lower potential (11). Only T left.

Shortest path following back from T is: $SBCDT$.

2 Marks per step:1 Mark per step in Table + 1 mark for steps description

10 Marks TOTAL — unseen problem

Q5. *(a) A card is drawn at random from a regular pack of 52 playing cards: What is the probability that it is not a picture card of any suit?* *[2]*

There are 12 picture cards in a pack: (Jack, Queen, King)x 4 suits.
Therefore there are 40 **non-picture** cards.
So probability of a non-picture card is:

$$P(\text{non} - \text{picture card}) = \frac{40}{52} = \frac{10}{13}$$

2 Marks — unseen problem

(b) Two cards are drawn at random from a regular pack of 52 playing cards:

(i) What is the probability that they are a pair of aces? *[2]*

Probability of drawing first ace is

$$P(\text{first ace}) = \frac{4}{52} = \frac{1}{13}$$

Probability of drawing second ace is

$$P(\text{second ace}) = \frac{3}{51}$$

SO probability of drawing a pair of aces is:

$$P(\text{drawing a pair of aces}) = \frac{1}{13}.\frac{3}{51}$$

2 Marks — unseen problem

(ii) What is the probability that they are any pair? *[2]*

Probability of drawing first card is

$$P(\text{first card}) = 1$$

Probability of drawing second card same number as first is

$$P(\text{second card same number}) = \frac{3}{51}$$

So probability of drawing any pair is:

$$P(\text{drawing any pair}) = \frac{3}{51}$$

2 Marks — unseen problem

(iii) What is the probability that they are the same suit? *[2]*

Probability of drawing first card is

$$P(\text{first card}) = 1$$

Probability of drawing second card same suit as first is

$$P(\text{second card same suit}) = \frac{12}{51}$$

So probability of drawing any pair is:

$$P(\text{drawing pair same suit}) = \frac{12}{51}$$

2 Marks — unseen problem

(iv) What is the probability that they are both club suit cards? *[2]*

Probability of drawing first ace is $P(\text{first card club}) = \frac{1}{4}$

Probability of drawing second card same club suit as first is

$$P(\text{second card another club}) = \frac{12}{51}$$

So probability of drawing a pair of clubs is:

$$P(\text{drawing pair same club suit}) = \frac{1}{4}.\frac{12}{51} = \frac{3}{51}$$

2 Marks — unseen problem

(c) *A fair coin and fair dice are thrown together. What is the probability of a head* or *a number greater than 3 being obtained?* *[3]*

Probability of getting head is

$$P(H) = \frac{1}{2}$$

Probability of getting a number greater than 3 is

$$P(> 3) = \frac{1}{2}$$

So probability of a head **and** a number greater than 3 is

$$P(H \text{ and } > 3) = P(H \cap > 3) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$

Now by the *addition rule*:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

.
So:

$$P(H\text{or} > 3) = P(H\cup > 3) = P(H)+P(> 3)-P(H\cap > 3) = \frac{1}{2}+\frac{1}{2}-\frac{1}{4} = \frac{3}{4}$$

3 Marks — unseen problem

13 Marks for TOTAL Question

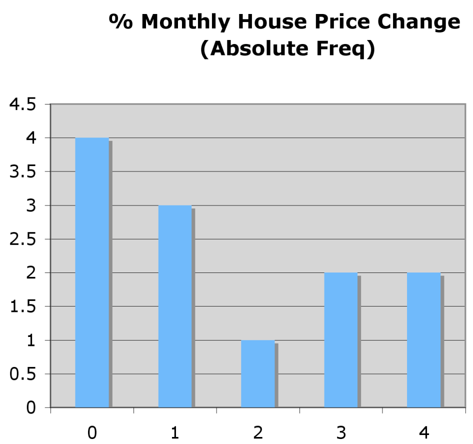
Q6. Consider a sample of size 12 about the monthly change in house prices.

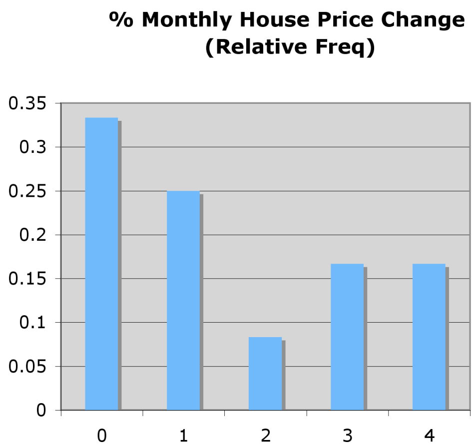
$$0\%,\ 4\%,\ 3\%,\ 3\%,\ 4\%,\ 1\%,\ 0\%,\ 2\%,\ 0\%,\ 1\%,\ 1\%,\ 0\%$$

Calculate the absolute *and* relative frequency *of each monthly change and draw a vertical bar graph for the sample.*

value a_j	absolute frequency n_j	relative frequency r_j
0%	4	0.333
1%	3	0.25
2%	1	0.083
3%	2	0.167
4%	2	0.167

Vertical Bar Graph: (Either Absolute or Relative frequency plot is adequate)





3 Marks each for absolute frequency n_j, relative frequency r_j and 2 marks for graph plot

8 Marks for TOTAL Question — unseen problem

Q7. Consider the following sample.

$$0,\ 2,\ 6,\ 1,\ 8,\ 4,\ 3,\ 6,\ 5,\ 4,\ 1,\ 2,\ 4,\ 5,\ 2$$

(a) Calculate the arithmetic mean $\bar{x}$ *and the* sample variance s^2.

There are **15 data samples**

arithmetic mean:

$$\begin{aligned}
\bar{x} &= (0+2+6+1+8+4+3+6+5+4+1+2+4+5+2)/15 \\
&= 53/15 \\
&= 3.533
\end{aligned}$$

2 Marks for arithmetic mean

The sample *variance s^2* is defined as

$$s^2 := \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

but also

$$s^2 := \frac{1}{n-1}\Big(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\Big)$$

which is easier to compute,

So

$$\begin{aligned}
s^2 &= \frac{1}{14}((0^2+2^2+6^2+1^2+8^2+4^2+3^2+6^2+5^2+4^2+1^2+2^2+4^2+5^2+2^2) \\
&\quad -15 \times 3.5333^2) \\
&= \frac{1}{14}(257 - 187.263) \\
&= \frac{69.737}{14} \\
&= 4.981
\end{aligned}$$

3 Marks for Variance

5 Marks Total — unseen problem

(b) *Calculate the* inter-quartile range IQR *and the* median x_{med} *of the sample.*

Sort Data:

$$x = \{0,\ 1,\ 1,\ 2,\ 2,\ 2,\ 3,\ 4,\ 4,\ 4,\ 5,\ 5,\ 6,\ 6,\ 8\}$$

Recap:

The **inter-quartile-range** is defined as the difference of the upper quartile and the lower quartile , i.e.

$$IQR := x_{0.75} - x_{0.25}$$

$$x_\alpha = \begin{cases} x_{[n\alpha+1]} & \text{if } n\alpha \text{ is not an integer} \\ \frac{x_{n\alpha}+x_{n\alpha+1}}{2} & \text{if } n\alpha \text{ is an integer} \end{cases}$$

$[n\alpha + 1] =$ the nearest integer to $n\alpha + 1$ which is lower or equal to $n\alpha$.

- the 0.25-quantile — **the lower quartile**,
- the 0.5-quantile — **the median**
- and the 0.75-quantile — **the upper quartile**.

In our case 0.25×15, 0.5×15 and 0.75×15 is not an integer.

So we get

- the 0.25-quantile — $x_4 = 2$
- the median – $x_8 = 4$
- and the 0.75-quantile — $x_{12} = 5$

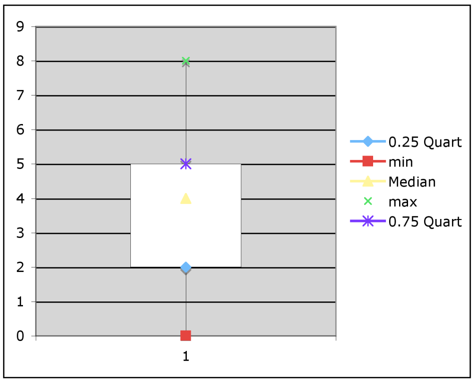
So $IQR = 5 - 2 = 3$ and the median is 4.

3 Marks for IQR, 1 Mark for Median

4 Marks Total — unseen problem

(c) Draw a box-plot *for the sample. Are there any* outliers*?*

Box Plot:



3 Marks

Outliers:
We say that a value x_i is an *outlier* if:

$$x_i > x_{0.75} + 1.5 \times IQR := z_u$$

or if

$$x_i < x_{0.25} - 1.5 \times IQR := z_l$$

So in our case
we have an outlier if:

$$x_i > 5 + 1.5 \times 3 = 9.5$$

or if

$$x_i < 2 - 1.5 \times 3 = -2.5$$

So we have **NO** OutlierS in THIS data sample.
3 Marks
6 Marks Total — unseen problem

15 Marks for TOTAL Question

Q8. *Given the following vectors:*

$$\mathbf{v} = (-1, 4), \mathbf{w} = (2, 3)$$

(a) What are the norms *of $\vec{v}$ and $\vec{w}$?*

Norm of $\|v\| = \sqrt{-1^2 + 4^2} = \sqrt{1 + 16} = \sqrt{17} = 4.12$
Norm of $\|w\| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13} = 3.61$

2 Marks — unseen problem

(b) What is the scalar product *$\vec{v}.\vec{w}$?*

$v.w = -1 \times 2 + 4 \times 3 = -2 + 12 = 10.$
2 Marks — unseen problem

(c) What is the angle θ between $\vec{v}$ and $\vec{w}$?

$$v.w = \|v\| \|w| \cos(\theta)$$

so

$$\cos(\theta) = \frac{v.w}{\|v\| \|w|}$$

from (a) and (b)

$$\cos(\theta) = \frac{10}{4.12 * 3.61} = 0.672$$

So $\theta = \cos^{-1}(0.732) = 47.75°$

3 Marks — unseen problem

(d) What is the vector cross product *$\vec{v} \times \vec{w}$?*

Let $n = 2$. We define the vector product of $v, w \in \mathbb{R}^2$ as a map $\times$: $\mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ with

$$v \times w = v_1 w_2 - v_2 w_1$$

So for

$$\mathbf{v} = (-1, 4), \mathbf{w} = (2, 3)$$

we get $v \times w = -1 * 3 - 4 * 2 = -3 - 8 = -11$

4 Marks — unseen problem

(e) *What is the area of the parallelogram spanned by* $\mathbf{v}$ *and* $\mathbf{w}$*?*
Area of parallelogram is $|\vec{v} \times \vec{w}| = 11$
2 Marks — unseen problem

13 Marks for TOTAL Question

Q9. Calculate the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -3 \\ 2 & 0 & 1 \end{pmatrix}$$

Can do determinant decomposition by any row (or column). As there is a zero in third row. We can exploit this to work out determinant more easily:

$$\begin{aligned}
\det A &= \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & -3 \\ 2 & 0 & 1 \end{vmatrix} \\
&= 2 \times \begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} - \mathbf{0} \times \begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\
&= 2 \times (2. - 3 - 3.1) + 1 \times (1.1 - 1.2) \\
&= 2 \times -9 + 1 \times -1 \\
&= -18 - 1 \\
&= -19
\end{aligned}$$

7 Marks — unseen problem

7 Marks for TOTAL Question

END OF EXAMINATION