On the spectral properties of the Brown-Ravenhall operator

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The fact that the Dirac is unbounded below creates problems if it is used to describe multi-particle relativistic systems since the resulting operator has a spectrum which covers the whole of the real line. To overcome this difficulty Brown and Ravenhall proposed the following one-particle model. To describe an electron in the field of its nucleus and subject to relativistic effects, the operator of Brown and Ravenhall is

$$\mathbf{B} := \Lambda_+\left(D_0 - \frac{e^2 Z}{|\cdot|}\right)\Lambda_+. \tag{1}$$

acting in the Hilbert space $\mathcal{H} := \Lambda_+(\mathrm{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^4)$. The notation in (1) is as follows

- D_0 is the free Dirac operator

  $$D_0 = c\boldsymbol{\alpha}\cdot \frac{\hbar}{i}\,\nabla + mc^2\beta \equiv \sum_{j=1}^{3} c\frac{\hbar}{i}\alpha_j \frac{\partial}{\partial x_j} + mc^2\beta,$$

  where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β are the Dirac matrices given by

  $$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix}$$

  with $0_2, 1_2$ the zero and unit 2×2 matrices respectively and σ_j the Pauli matrices

  $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Λ_+ denotes the projection of $\mathrm{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ onto the positive spectral subspace of D_0, that is $\chi_{(0,\infty)}(D_0)$, where $\chi_{(0,\infty)}$ is the characteristic function of $(0, \infty)$. If we set

  $$\widehat{f}(\mathbf{p}) \equiv \mathcal{F}(f)(\mathbf{p}) = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \int_{\mathbb{R}^3} e^{-i\,\mathbf{x}\cdot\mathbf{p}/\hbar} f(\mathbf{x})\,\mathrm{d}\,\mathbf{x}$$

for the Fourier transform of f, then it follows that

$$(\Lambda_+ f)^\wedge(\mathbf{p}) = \Lambda_+(\mathbf{p})\widehat{f}(\mathbf{p}),$$

where

$$\Lambda_+(\mathbf{p}) = \frac{1}{2} + \frac{c\boldsymbol{\alpha}\cdot\mathbf{p} + mc^2\beta}{2\mathbf{e}(p)}, \quad \mathbf{e}(p) = \sqrt{c^2p^2 + m^2c^4} \tag{2}$$

with $p = |\mathbf{p}|$.

- $2\pi\hbar$ is Planck's constant, c the velocity of light, m the electron mass, $-e$ the electron charge, and Z the nuclear charge.

The lecture will discuss spectral properties of operators $b_{l,s}$ appearing in the partial wave decomposition of **B**: the indices l, s, denote the angular momentum channel and spin respectively. The following topics will be covered: the value of the critical charge $Z_c(l, s)$ which yields the positivity of $b_{l,s}$, the charge range for essential self-adjointness, and the charge range for the absence of embedded eigenvalues.