NUMERICAL UNIQUENESS AND EXISTENCE THEOREM FOR SOLUTION OF LIPPMANN-SCHWINGER EQUATION TO TWO DIMENSIONAL SOUND SCATTERING PROBLEM

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In this paper, we are concerned with the following scattering problem for the Helmholtz equation in the inhomogeneous media: find $u : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\triangle u(x) + \kappa^2 b(x) u(x) = 0, \;\; x \in \mathbb{R}^2 \tag{1}$$

$$u = u^i + u^s, \tag{2}$$

$$\lim_{r=|x|\to\infty} r^{1/2}\left(\frac{\partial u^s}{\partial r} - \mathbf{i}\kappa u^s\right) = 0. \tag{3}$$

Here, $\mathbf{i} = \sqrt{-1}$ and κ is the wave number of the incidence wave. It is well known that this problem is given as a mathematical modeling of a two dimensional sound scattering problem [1].

We here assume the following:

Assumption 1. *The refractive index $b : \mathbb{R}^2 \to \mathbb{C}$ is a given smooth function and*

$$a(x) = b(x) - 1 \tag{4}$$

has a compact support such that

$$\mathbf{supp}\; a \subset \overline{B}_\rho, \quad B_\rho = \{x \in \mathbb{R}^2 : |x| < \rho\}. \tag{5}$$

Furthermore, we assume that

$$a \in W^{\mu,2}(\mathbb{R}^2) \tag{6}$$

with $\mu > 1$.

Here, the Sobolev space $W^{\mu,2}(\mathbb{R}^2)$ consists of functions $w \in L^2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} (1 + |\xi|^2)^\mu |\hat{w}(\xi)|^2 d\xi < \infty, \tag{7}$$

where

$$\hat{w}(\xi) = \int_{\mathbb{R}^2} \mathbf{e}^{-\mathbf{i}x\cdot\xi} w(x)dx, \ \xi \in \mathbb{R}^2 \tag{8}$$

is the Fourier transform of w.

The incidence wave is assumed to be

$$u^i(x) = \exp\left(\mathbf{i}\kappa d \cdot x\right) \tag{9}$$

with a fixed $d \in \mathbb{R}^2$, $|d| = 1$. The incidence wave u^i is a plane wave solution of the Helmholtz equation

$$\triangle u(x) + \kappa^2 u(x) = 0. \tag{10}$$

The scattering wave u^s for this incidence wave is assumed to satisfy the Sommerfeld radiation condition (3).

It is known [1] that the problem defined by (1)-(3) is equivalent to the Lippmann-Schwinger equation

$$u(x) = \kappa^2 \int_{\mathbb{R}^2} E_\kappa(x-y)a(y)u(y)dy + u^i(x), \ x \in \mathbb{R}^2, \tag{11}$$

where $E_\kappa(x) = (\mathbf{i}/4)H_0^{(1)}(\kappa|x|)$. Here, $H_0^{(1)}$ is the Hankel function of the first kind of order zero. A fast solver for this equation has been proposed and studied mathematically in detail by Vainikko [2]. The monograph Saranen-Vainikko [3] has described Vainikko's theory in detail with its mathematical background. According to [2] and [3], first, we somewhat simplify and generalize (11). If we scale the independent variables x and y by $\tilde{x} = \kappa x$ and $\tilde{y} = \kappa y$, respectively, without loss of generality, we can assume that $\kappa = 1$. Further, instead of u^i, which is a solution of homogenized Helmholtz equation with $b(x) = 1$ for all $x \in \mathbb{R}^2$, we considier a general function f. We assume the following:

Assumption 2. $f \in W^{\mu,2}_{\mathbf{loc}}(\mathbb{R}^2), \mu > 1$.

Let us recall that $a \in W^{\mu,2}(\mathbb{R}^2)$ with **supp** $a \subset \overline{B}_\rho = \{x \in \mathbb{R}^2 : |x| \leqq \rho\}$. Then, the problem can be formulated as: find $u \in C(\overline{B}_\rho)$ satisfying the following equation

$$u(x) = \int_{B_\rho} E_1(x-y)a(y)u(y)dy + f(x), \ x \in B_\rho. \tag{12}$$

By the Sobolev embedding theorem, $W^{\mu,2}_{\mathbf{loc}} \subset C(\mathbb{R}^2)$ for $\mu > 1$. Multiplying both sides of (12) by $a(x)$, we can rewrite the equation with respect to $v(x) = a(x)u(x)$ as an unknown function:

$$v(x) = a(x) \int_{B_\rho} E_1(x-y)v(y)dy + a(x)f(x), \ x \in B_\rho. \tag{13}$$

A crucial observation is that, for $x \in \overline{B}_\rho$, only the values from $\overline{B}_{2\rho}$ of $E_1(x)$ are involved in the integral, *i.e.*, changing E_1 outside this ball, the solution $v(x)$ does not change in $\overline{B}_\rho$. We exploit this observation and redefine the kernel E_1 in $G_R \backslash \overline{B}_R$ where

$$G_R = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1| < R, |x_2| < R\}$$

is an open box and $R > 2\rho$ is a parameter. Namely, we define

$$\mathcal{K}(x) = \begin{cases} E_1(x) = \dfrac{\mathbf{i}}{4} H_0^{(1)}(|x|), & |x| \leqq R \\ 0, & x \in G_R \backslash \overline{B}_R \end{cases} \tag{14}$$

After that we extend $\mathcal{K}, a, af$ and v from G_R to $\mathbb{R}^2$ as $2R$-periodic functions with respect to x_1 and x_2. Thus, we have arrived at the biperiodic integral equation

$$v(x) = a(x) \int_{G_R} \mathcal{K}(x - y) v(y) dy + a(x) f(x), \tag{15}$$

which is equivalent to (12).

The main theorem in [2] and [3] is the following:

Theorem 1 (Vainikko)**.** *Assume that functions a and f satisfies Assumptions 1, 2 mentioned above and the homogeneous equation corresponding to (15) has in H^0 only the trivial solution. Then (15) has a unique solution $v \in H^\mu$ has a unique solution $v_N \in \mathcal{T}_N$ for sufficiently large N, and*

$$\|v_N - v\|_\lambda \leqq c N^{\lambda - \mu} \|v\|_\mu, \quad (0 \leqq \lambda \leqq \mu). \tag{16}$$

In [1], the existence of a unique solution in a function space of continuous functions is proved for the Lippmann-Schwinger equation in $\mathbb{R}^3$ using the unique continuation principle.

In this paper, for the sake of simplicity, we treat the case of $\mu \geqq 3/2$ and present Theorem giving a sufficient condition of guaranteeing that the homogeneous equation corresponding to (15) has in H^0 only the trivial solution. Thus, it can be seen as an another uniqueness theorem. A remarkable feature of this theorem is that the sufficient condition shown in this theorem can be evaluated by verified numerical computations. Furthermore, if such a sufficient condition holds, it also gives an upper bound of $\|(I - aK)^{-1}\|_{\mathcal{L}(H^0)}$ and a tight error bound between the exact solution v and an approximate solution $\tilde{v}$ which is generated by computer. As a result, this paper is to present a numerical uniqueness and existence theorem for (15), which asserts the existence of a unique solution around an approximate solution computed by numerical calculation.

References

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