DIRICHLET AND NEUMANN DENSITIES FOR THE SCHRÖDINGER OPERATOR IN ONE DIMENSION

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We consider the one-dimensional Schrödinger equation $-y'' + qy = \lambda y$, where $q = q(x)$ is real valued, locally integrable over the interval $[0, \infty)$, and the spectral parameter λ is real. For fixed (real) λ belonging to the essential support of the absolutely continuous part of the spectrum, let Γ_λ denote the real linear space spanned by the squares of all solutions $y(\cdot, \lambda)$ for y. We may regard functions $R(\cdot, \lambda)$ belonging to Γ_λ as the third component of a triple of funtions P, Q, R satisfying a particular set of simultaneous first order linear equations in the coordinate x. These first order equations may be used to define an invariant (i.e. independent of the coordinate x) scalar product, as well as a vector product , of any pair of functions in the space Γ_λ.

Given a pair of functions $R_1(\cdot, \lambda), R(\cdot, \lambda)$ in Γ_λ, we shall use the algebraic properties of this space, as well as results from the theory of value distribution, to define a pair of absolutely continuous measures μ_1, μ_2 as weak limits of the ratio R_1/R and its square. Denoting by m_1, m_2 the density functions for the measures μ_1, μ_2 respectively, one can derive totally explicit algebraic formulae for the Dirichlet and Neumann densities of the Schrödinger operator in terms of m_1 and m_2.

As an example of results of this kind, define solutions $u(\cdot, \lambda), v(\cdot, \lambda)$ of the Schrödinger equation to satisfy the standard initial conditions $u = 1, v = 0, u' = 0, v' = 1$ at $x = 0$, and define the measures $\mu_{1,}\mu_2$ by

$$\mu_1(\Sigma) = \lim_{x \to \infty} \int_\Sigma \frac{R_1(x, \lambda)}{R(x, \lambda)} d\lambda \tag{1}$$

$$\mu_2(\Sigma) = \lim_{x \to \infty} \int_\Sigma \frac{R_1(x, \lambda)^2}{R(x, \lambda)^2} d\lambda \tag{2}$$

where $R_1 = u^2 - v^2$ and $R = u^2 + v^2$.

Then, in terms of densities m_1, m_2 for the measures μ_1, μ_2 respectively, the spectral density M_d for the Dirichlet Schrödinger operator is given as a function of λ by

$$\pi M_d = \frac{m_2 - m_1^2}{1 + m_1^2 - m_1 - m_2} \tag{3}$$

with a similar formula for the Neumann density.

These formulae for Dirichlet and Neumann density as functions of m_1 and m_2 are invertible to give m_1 and m_2 as functions of Dirichlet and Neumann densities.