EPSRC Spectral Theory Network Conference I

Cardiff University

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Timetable

Friday 19 Oct, 2001

8:50	**B.M. Brown** - Introductory remarks
9:00	**M. Marletta** (Leicester) - The Spectrum of the Ekman Flow and Related Block-Operator Problems
10:00	**A. Pushnitski** (Loughborough) - Trace Formulae for Stark Operator
11:00	*Tea/Coffee*
11:30	**C. Bennewitz** (Lund, Sweden) - Inverse Spectral Theory for Left Definite Sturm-Liouville Equations
12:30	*Lunch*
13:30	**M. Levitin** (Heriot-Watt) - Computational Experiments and Bounds on Low Eigenvalues of the Dirichlet Laplacian for Planar Domains
14:30	**A. Spence** (Bath) - Inverse Iteration and Inexact Solves
15:30	*Tea/Coffee*
16:00	**B. Straughan** (Durham) - Spectral Methods for Calculating Eigenvalues in Porous-Fluid Convection Problems.
17:00	**F. Rofe-Beketov** (Kharkov, Ukraine) - Inverse Spectral Problems for Non-selfadjoint Differential Operators on the Half Axis by Generalized Spectral Matrix
18:30	*Dinner*

Saturday 20 Oct, 2001

9:00	**D.B. Pearson** (Hull) - Recent Developments in Value Distribution Theory for Solutions of the Schrödinger Equation
10:00	**E.B. Davies** (Kings, London) - Computing Thresholds of Schrödinger Operators
11:00	*Tea/Coffee*
11:30	**V.I. Burenkov** (Cardiff) - Spectral Stability of the Neumann Laplacian
12:30	*Lunch*
13:30	**R. Ashurov** (Tashkent, Uzbekistan) - On summability of spectral resolutions, connected with differential and pseudo-differential operators
14:30	**Y. Netrusov** (Bristol) - Estimates of s–numbers of differential operators
15:30	*Tea/Coffee*
16:00	**W.D. Evans** (Cardiff) - Non-self-adjoint Sturm-Liouville operators and Hamiltonian systems

Abstracts

On summability of spectral resolutions connected with differential and pseudo-differential operators

R.R. Ashurov
National University of Uzbekistan, Tashkent

Let Ω be an arbitrary domain from $\mathbb{R}^N$ and A be an elliptic, symmetric, non-negative pseudo-differential operator (PDO) acting in $L_2(\Omega)$ with the definition domain $C_0^\infty(\Omega)$.

Let $\{E_\lambda\}$ be identity resolution of an arbitrary self-adjoint extension $\widehat{A}$ of the operator A in $L_2(\Omega)$. By Gårding's theorem E_λ is an integral operator with the kernel $\theta(x, y, \lambda)$, which is called the spectral function, i.e.

$$E_\lambda f(x) = \int_\Omega \theta(x, y, \lambda) f(y) dy, \quad f \in L_2(\Omega). \tag{1}$$

The expression $E_\lambda f(x)$ is called the spectral resolution of the element $f \in L_2(\Omega)$. If $\widehat{A}$ has purely point spectrum, then (1) coincides with partial sums of Fourier series on eigenfunctions of the operator $\widehat{A}$.

Denote by $E_\lambda^s f$ the Riese means of order $s \geq 0$ for the spectral resolution $E_\lambda f$, i.e.

$$E_\lambda^s f(x) = \int_0^\lambda \left(1 - \frac{\lambda}{t}\right)^s dE_t f(x). \tag{2}$$

Note that $E_\lambda^0 = E_\lambda$.

We are going to investigate the convergence almost everywhere (a.e.) of Riece means $E_\lambda^s f$ to f, as $\lambda \to +\infty$.

In case of an arbituary elliptic differential operator A, the general result belongs to L. Hörmander (1969). He proved, that if $s > 2(N-1)\left(\frac{1}{p} - \frac{1}{2}\right)$ and $f \in L_p(\Omega)$, $1 \leq p \leq 2$, then

$$\lim_{\lambda \to \infty} E_\lambda^s f(x) = f(x) \quad \text{a.e. on } \Omega. \tag{3}$$

If the leading symbol of the elliptic operator is constant then, as it was proved by R.R. Ashurov in 1992, the last result can be improved as $s > N\left(\frac{1}{p} - \frac{1}{2}\right)$.

For the Laplace operator this result was obtained by Sh. A. Alimov (1971).

We note that if $0 \leq s < N\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}$, then there exists a function $f \in L_p(\Omega)$, such that

$$\overline{\lim}_{\lambda \to \infty} |E_\lambda^s f(x)| = +\infty$$

in a set with positive measure. For the Laplace operator this result was proved by V.A. Il'in and Sh. A. Alimov (1972) and by R.R. Ashurov (1992) for an arbitrary elliptic differential operator.

For the first time A. Bastis (1983) started to investigate convergence a.e. of spectral resolutions $E_\lambda f$, corresponding to an arbitrary elliptic differential operator, to smooth functions and proved that

$$\lim_{\lambda \to \infty} E_\lambda f(x) = f(x) \quad \text{a.e. on } \Omega,$$

for functions from the Liouville class $L_p^a(\Omega)$, if $a > N\left(\frac{1}{p} - \frac{1}{2}\right)$.

For the elliptic differential operator with constant coefficients and definition domain $C_0^\infty(\mathbb{R}^N)$, i.e. for multiple Fourier integrals, it is possible to state a more precise result:

Theorem 1 (R. Ashurov, K. Buvaev, 1998). *Let* $a > (N-1)\left(\frac{1}{p} - \frac{1}{2}\right)$, $1 \le p \le 2$. *Then for a function* $f \in L_p^a(\mathbb{R}^N)$ *we have*

$$\lim_{\lambda\to\infty} E_\lambda f(x) = f(x) \quad \textit{a.e. on } \mathbb{R}^N.$$

The next theorem shows that theorem 2 can not be improved at least for $p = 1$.

Theorem 2 (R. Ashurov, K. Buvaev, 1997). *Let* $N \ge 2$, $0 \le a < \frac{N-1}{2}$. *Then there exists a function* $f \in L_1^a(\mathbb{R}^N)$, *such that*

$$\overline{\lim}_{\lambda\to\infty} |E_\lambda f(x)| = +\infty, \quad x \in E,$$

where E *is a set with positive measure.*

The question concerning the sharpness of A. Bastis's theorem still remains open. But if we consider a class of pseudo-differential operators, then it is possible to see that positive and negative theorems coinside.

Theorem 3 (R. Ashurov, 2000). *Let* $\{E_\lambda\}$ *be the identity resolution of an arbitrary self-adjoint extansion in* $L_2(\Omega)$ *of the elliptic, symmetric, non-negative PDO. Then for an arbitrary function f from the Liouville class* $L_p^a(\Omega)$, $1 \le p \le 2$, $a > N\left(\frac{1}{p} - \frac{1}{2}\right)$, *we have*

$$\lim_{\lambda\to\infty} E_\lambda f(x) = f(x) \quad \textit{a.e. on } \Omega.$$

Theorem 4 (R. Ashurov, 2000). *Let* $a < N\left(\frac{1}{p} - \frac{1}{2}\right)$. *Then there exists an elliptic PDO with constant coefficients and definition domain* $C_0^\infty(\mathbb{R}^N)$, *a function* $f \in L_p^a(\mathbb{R}^N)$, $1 \le p \le 2$, *such that*

$$\overline{\lim}_{\lambda\to\infty} |E_\lambda^s f(x)| = +\infty,$$

in a set with positive measure.

Inverse spectral theory for left-definite Sturm-Liouville equations

Christer Bennewitz
Lund University, Sweden

We consider the equation $-(pu')' + qu = \lambda wu$ on an interval $[0, b)$ where $1/p$ and q are positive and locally integrable, and w is locally integrable but not necessarily of one sign. There is a spectral theory for this equation in a Hilbert space $\mathcal{H}$ with inner product $\langle u, v\rangle = \int_0^b pu'\overline{v'} + qu\overline{v}$. Specifically, introducing appropriate separated boundary conditions there is a selfadjoint operator T and a generalized Fourier transform $\mathcal{F} : \mathcal{H} \to L_\rho^2$ diagonalizing T. Here L_ρ^2 is the Hilbert space of square integrable functions with respect to the *spectral measure* $d\rho$.

We shall answer the following question: *To what extent is the operator* T, i.e. *the interval* $[0, b)$, *the coefficients* p, q *and* w, *and the boundary conditions determined by the spectral measure* $d\rho$?

This question has some importance in connection with the so called Camassa-Holm equation for shallow water waves, which is intimately connected with a spectral problem of the type above.

Spectral stability of the Neumann Laplacian

V.I. Burenkov [1]
Cardiff University
E.B. Davies
Kings College London

We prove the equivalence of Hardy- and Sobolev-type inequalities, certain uniform bounds on the heat kernel and some spectral regularity properties of the Neumann Laplacian associated with an arbitrary region of finite measure in Euclidean space. We also prove that if one perturbs the boundary of the region within a uniform Hölder category then the eigenvalues of the Neumann Laplacian change by a small and explicitly estimated amount.

Let Ω be an arbitrary region in $\mathbf{R}^N$ and let us define the Neumann Laplacian $-\Delta_N$ to be the non-negative self-adjoint operator acting in $L^2(\Omega)$ and associated with the quadratic form $\int_\Omega |\nabla f|^2 \,\mathrm{d}x$. Assume that $-\Delta_N$ has compact resolvent and denote by $\lambda_n, n = 0, 1, 2, ...$ the eigenvalues of $-\Delta_N$ written in increasing order and repeated according to their multiplicity. The main result is as fol lows.

Theorem. Let $N \geq 2, 0 < \gamma \leq 1, M, \delta > 0$ and integer $s \geq 1$. Moreover, let $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ a family of rotations. Suppose that $\Omega_1 \subset \mathbf{R}^N$ is a bounded region such that

$$\partial\Omega_1 \in \mathrm{Lip}\,(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s).$$

Then for every integer $n \geq 1$ there exist $b_n = b_n(\Omega_1), \varepsilon_n = \varepsilon_n(\Omega_1) > 0$ such that for all $0 < \varepsilon \leq \varepsilon_n$ and for all bounded regions Ω_2, for which

$$\partial\Omega_2 \in \mathrm{Lip}\,(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$$

and $\{x \in \Omega_1 : \mathrm{dist}\,(x, \partial\Omega_1) > \varepsilon\} \subset \Omega_2 \subset \Omega_1$, the inequality

$$(1 - b_n\varepsilon^\gamma)\lambda_{n,1} \leq \lambda_{n,2} \leq (1 + b_n\epsilon^\gamma)\lambda_{n,1}$$

holds.

Here $\mathrm{Lip}\,(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is the standard class of bounded regions whose boundaries locally satisfy the Lipschitz condition with the exponent γ and constant M. It is important that for Ω_1 and Ω_2 the parameters defining this class are the same.

[1] Supported by INTAS (grant 99-01080) and RFBR (grant 99-01-00834).

Computing thresholds of Schrödinger operators

E.B. Davies [2]
Kings College London

We consider the exact values of t (called thresholds) at which the number of negative values of a Schrödinger operator $H = -\Delta + tV$ changes. Assuming that H acts in $L^2(\mathbf{R}^n)$ and that $V(x) \to 0$ as $|x| \to \infty$ this is a difficult problem because 0 is the bottom of the continuous spectrum of H so rigorous two-sided bounds on small negative eigenvalues are not easy to obtain. The method involves transferring to a new operator which has different continuous spectrum but the same number of negative eigenvalues.

Non-self adjoint differential operators and Hamiltonian systems

W.D. Evans
Cardiff University

In 1957 A.R.Sims published an extension of the Weyl limit-point/limit- circle theory to second-order differential equations with a complex potential. Further results on this problem were obtained by Birger and Kalyabin in 1976, and, in 1999, by Brown, McCormack, Evans and Plum. In this recent work, the three cases in the classification of Sims are characterised by the finiteness of a Sobolev-type norm of solutions when the spectral parameter lies outside a specified set. The lecture will briefly outline these results before going on to describe recent work of Brown, Evans and Plum on the analogous problem for general Hamiltonian systems.

Computational Experiments and Bounds on Low Eigenvalues of the Dirichlet Laplacian for Planar Domains

Michael Levitin
Heriot-Watt University

We give a brief survey of existing uniformal and isoperimetric estimates of the low eigenvalues of the Dirichlet Laplacian in planar domains. We proceed to discuss the extensive numerical experiments on the possible range of $(\lambda_2/\lambda_1, \lambda_3/\lambda_1)$, conducted jointly with R Yagudin, and present some propositions which may help to improve the existing estimates for the supremum of λ_3/λ_1.

[2] published in LMS J. Comp. Math. 2 (1999) 139-154

The Spectrum of the Ekman Flow and Related Block-Operator Problems

M. Marletta
University of Leicester

We consider block operator eigenvalue problems, of the form $Ax = \lambda Bx$. Problems of this form arise in numerous applications, most notably in MHD. We give some 'toy' examples to illustrate how pathological the spectra may be, and consider conditions under which these problems can be reduced to differential equation problems.

A particularly interesting example is the block operator problem describing the Ekman boundary layer. For this we locate the essential spectrum and develop a Titchmarsh-Weyl coefficient $M(\lambda)$. This allows a rigorous analysis of the regularizations of the Ekman problem, and in particular a proof of their spectral exactness.

This is joint work with Leon Greenberg of the University of Maryland.

Estimates of s–numbers of differential operators

Yuri Netrusov
University of Bristol

Given a linear differential operator T from L^2_m to L^2_{-m}. We consider conditions on T (sometimes under additional restrictions) such that the operator T is continuous or belongs to the Neumann–Schouten class S_r, $r \in (0, \infty)$.

Recent Developments in Value Distribution Theory for Solutions of the Schrödinger Equation

D.B.Pearson
University of Hull

Given a measurable function f from $\mathbb{R}$ to $\mathbb{R}$, the value distribution of f may be described in terms of the (Lebesgue) measure of the set of points in the domain of f at which f assumes values in prescribed Borel subsets of $\mathbb{R}$. If f is the boundary value of a Herglotz function, the value distribution of f is related to properties of associated spectral measures, and the idea of value distribution may be extended to cover Herglotz functions having complex boundary values.

The talk will describe recent results of the Hull spectral theory group in which value distribution theory is applied to the Weyl m-function for the one dimensional Schrödinger operator and used to derive precise estimates for asymptotic behaviour of solutions of the Schrödinger equation.

Particular applications will be a description of asymptotics on the support of the a.c. measure, an analysis of sparse potentials and their associated value distribution, and a link between spectral theory and asymptotics for the m-function and the geometry of hyperbolic space.An extension of value distribution theory in which Lebesgue measure is replaced by a more general Herglotz-type measure will also be discussed.

Trace formulae for Stark operator

Alexander Pushnitski
Loughborough University

In $L^2(\mathbb{R}^3, dx)$ we consider the unperturbed and perturbed Stark operators $H_0 = -\Delta + Ex$ and $H = H_0 + V$, where E is a constant vector (the strength of the electric field) and the perturbation potential V belongs to the Schwartz class. We prove the high energy asymptotic expansion for the (modified) perturbation determinant of this pair of operators. By a standard procedure, this expansion yields trace formulae of the Buslaev-Faddeev type.

The talk is based on a recent joint work with Evegeni Korotyaev (Humboldt University, Berlin).

Inverse Spectral Problems for Non-selfadjoint Differential Operators on the Half Axis by Generalized Spectral Matrix

F.S. Rofe-Beketov
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Eigenfunction expansion formulas for the boundary problem on the halfaxis

$$l[y] \equiv -y'' + q(x)y = \lambda y \qquad (0 \le x < \infty)$$

$$y'(0) - Ay(0) = 0$$

and the Parseval equality in the selfadjoint case (i.e. $q(x) \in \mathbb{R}, A \in \mathbb{R}$) have the form:

$$E_f(\lambda) = \int_0^\infty f(x)\,\omega(x,\lambda)\,dx,$$

$$f(x) = \int_{-\infty}^\infty E_f(\sqrt{\lambda}\,)\,\omega(x,\sqrt{\lambda})\,d\varrho(\sqrt{\lambda}),$$

$$\int_{-\infty}^{\infty} E_f(\sqrt{\lambda})\, E_g(\sqrt{\lambda})\, d\rho(\lambda) \;=\; \int_0^{\infty} f(x)\, g(x)\, dx$$

.

Here $\omega(x, \lambda)$ is a solution of the problem $l[\omega] = \lambda^2(\omega), \omega(0, \lambda) = 1, \omega^{'}(0, \lambda) = A$, $\rho(\lambda)$ is a spectral function of the problem. V.A.Marchenko introduced the generalization of these formulas for nonselfadjoint case, where $q(x) \in L^1_{loc}$ is an arbitrary complexvalued function and $A \in \mathbb{C}$. In this case ρ occurs a distribution on the topological space of even entire functions of exponential type. Above mentioned results are generalized by the speaker for the case of the boundary problem for the above mantioned Sturm–Liouville equation with the operator-valued potential $q(x) \in B(H)$ and $A \in B(H)$, where H is a separable Hilbert space. In this case ρ occurs the generalized spectral matrix, the entries of which are the distributions of Marchenko type. Inverse spectral problem by generalized spectral matrix (i.e. reconstruction of $q(x)$ and A) is solved with the help of integral equation of I.M.Gelfand–B.M.Levitan type. The necessary and sufficient conditions for the generalized spectral matrix ρ of the boundary-value problem with the n-time differentiable operator-valued potential $q(x)$ are obtained. For the selfadjoint case (i.e. when $q^*(x) = q(x), A^* = A$) a representation of the generalized spectral matrix with the help of the Stieltjes integral by nondecreasing operatorvalued function is obtained. Similar results for the case of the boundary condition $y(0) = 0$ are proved.

References

[1] V.A.Marchenko and F.S.Rofe-Beketov .*Expansion in eigenfunctions of nonselfadjoint singular differential operators . Dokl. Akad. Nauk SSSR, 1958, v. 120, No 5, 963–966 (in Russian)*

[2] F.S. Rofe-Beketov. Expansion in eigenfunctions of infinite systems of differential equations in the nonselfadjoint and selfadjoint cases, Mat. Sbornik, 1960, v. 51, No 3, 293–342 (in Russian)

[3] V.A. Marchenko. Sturm–Liouville Operators and Applications , Birkhauser Verlag, Basel –Boston – Stuttgart, 1986.

Inverse Iteration and Inexact Solves [3]

Alastair Spence
University of Bath

When steady solutions of complex physical problems are computed numerically it is often crucial to compute their *stability* in order to, for example, carry out a sensitivity analysis, or help understand complex nonlinear phenomena near a bifurcation point. Usually a stability analysis requires the solution of an eigenvalue problem which may arise in its own right or as an

[3] This is joint work with Jörg Berns-Müller, Ivan Graham and Eero Vainikko at Bath and Andrew Cliffe at AEA Technology. It is suppported by UK EPSRC Grant GR/M59075

appropriate linearisation. In the case of PDEs discretised by the finite element method, the corresponding matrix eigenvalue problem will often be of *generalised* form:

$$A\mathbf{x} = \lambda M\mathbf{x} \tag{1}$$

with A generally unsymmetric and M positive semi-definite. Only a small number of "dangerous" eigenvalues are usually required, often those (possibly complex) eigenvalues nearest the imaginary axis. In this context it is usually necessary to perform "inverse" iterations, which require repeated solution of systems of the form

$$(A - \sigma M)\mathbf{y} = M\mathbf{x}, \tag{2}$$

for some *shift* σ (which will probably be an approximation to an eigenvalue) and for various right-hand sides $\mathbf{x}$. In large applications systems (2) have to be solved iteratively, requiring "*inner iterations*".

In this talk we will describe recent progress in the construction, analysis and implementation of fast algorithms to compute the desired eigenvalues.

We discuss an application to the computation of bifurcations in Navier-Stokes problems discretised by mixed finite elements applied to the velocity-pressure formulation.

Parallel experiments illustrating performance on a Beowulf cluster will also be given.

Spectral Methods for Calculating Eigenvalues in Porous-Fluid Convection Problems

Brian Straughan
University of Durham

We examine the concepts of linearised instability and nonlinear energy stability in fluid mechanics. Eigenvalue problems from both the linear and nonlinear problems are derived and the connection (or lack of it) between the two is investigated. Spectral methods are then described for numerical solution of eigenvalue problems. We concentrate on a convection problem in porous media flow, multi-component convection in a fluid layer, atmospheric pollution studies and convective overturning, and if time permits, stability studies in a fluid overlying a porous layer will be addressed.