# First-order linear boundary value problems

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## 1. Abstract

This lecture reports on joint work with Anthippi Poulkou, Department of Mathematics, University of Athens.

The general Lagrange symmetric first-order differential equation with Lebesgue integrable coefficients, on the open interval $(a,b)$ of the real line $\mathbb{R}$, has the form, defining the differential expression $M[\cdot]$,

$$M[y](x) := i\rho(x)y'(x) + \tfrac{1}{2}i\rho'(x)y(x) + q(x)y(x) = \lambda w(x)y(x) \text{ for all } x \in (a,b)$$

where $\lambda \in \mathbb{C}$ is the complex spectral parameter. Here the coefficients $\rho, q, w$ satisfy the conditions

- $(i)$ $\rho, q, w : (a,b) \to \mathbb{R}$
- $(ii)$ $\rho \in AC_{\text{loc}}(a,b)$ and $\rho(x) > 0$ for all $x \in (a,b)$
- $(iii)$ $q, w \in L^1_{\text{loc}}(a,b)$
- $(iv)$ $w(x) > 0$ for almost all $x \in (a,b)$.

The right-definite spectral analysis for this differential equation takes place in the Hilbert function space $L^2((a,b);w)$ with norm and inner-product

$$\|f\|_w^2 := \int_I w\,|f|^2 \ \textit{and}\ (f,g)_w := \int_a^b w(x)f(x)\overline{g}(x)\,dx.$$

A necessary and sufficient condition to ensure that the differential expression $M[\cdot]$ generates a maximal operator in $L^2((a,b);w)$ with equal deficiency indices $d^{\pm} = 1$ whose self-adjoint restrictions have discrete spectra, is

$$\int_a^b \frac{w(x)}{\rho(x)}\,dx < +\infty.$$

With this condition satisfied the GKN boundary condition method can be applied to give symmetric boundary value problems with the following properties:

**Theorem 1.1.** *Let $T$ be a self-adjoint restriction of the maximal operator generated by $M[\cdot]$; then $T$ has the following spectral properties*:

$(i)$ *The spectrum $\sigma(T)$ of $T$ in $L^2((a,b);w)$ is simple and discrete.*

$(ii)$ *The spectrum $\sigma(T)$ is unbounded above and below on $\mathbb{R} \subset \mathbb{C}$, and so may be denoted by, here $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$,*

$$\sigma(T) = \{\lambda_n \in \mathbb{R} : n \in \mathbb{Z}\}$$

*with*

$$\lambda_n < \lambda_{n+1} \text{ for all } n \in \mathbb{Z}, \text{ and } \lim_{n \to \pm\infty} \lambda_n = \pm\infty.$$

$(iii)$ *There exists a positive number $k > 0$, with*

$$k = 2\pi \left( \int_a^b \frac{w(x)}{\rho(x)}\, dx \right)^{-1},$$

*such that*

$$\lambda_{n+1} - \lambda_n = k \text{ for all } n \in \mathbb{Z}.$$

$(iv)$ *There exists an entire $(integral)$ function $\varphi : \mathbb{C} \to \mathbb{C}$, generated by the boundary value problem, with the properties*

$(i)$ $\varphi(\lambda) = 0$ *if and only if* $\lambda \in \{\lambda_n : n \in \mathbb{Z}\}$
$(ii)$ $\varphi'(\lambda_n) \neq 0$ *for all* $n \in \mathbb{Z}$.

## 2. Kramer analytic kernels

The boundary value problems discussed in Section 1 generate Kramer analytic kernels in the Hilbert space $L^2((a,b);w)$.

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