**Meeting on Computational and Analytic Problems in Spectral Theory**

Cardiff School of Computer Science & Informatics, June 6th-9th 2016

# Abstracts of Talks

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## A VERSION OF THE RADÓ-KNESER-CHOQUET THEOREM FOR SOLUTIONS OF THE HELMHOLTZ EQUATION IN 3D

Giovanni Alberti

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In this talk I will discuss a method to prove the absence of critical points for the Helmholtz equation in 3D. The key element of the approach is the use of multiple frequencies in a fixed range, and the proof is based on the spectral analysis of the associated problem. This question is strictly connected with the Radó-Kneser-Choquet theorem, whose direct extension to the Helmholtz equation or to three dimensions is not possible.

## STABILITY OF STATIONARY SOLUTIONS FOR REACTION–DIFFUSION–FLOWS ON FINITE METRIC GRAPHS

Joachim von Below

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The Lyapunov stability of stationary solutions for the flow governed by reaction–diffusion–equations on the edges of a metric graph $G$ will be discussed under continuity and weighted Kirchhoff transition conditions at the vertices:

$$
(1) \qquad \begin{cases} u \in \mathcal{C}(G \times [0,\infty)) \cap \mathcal{C}_K^{2,1}(G \times (0,\infty)), & \\ \partial_t u_j = \partial_j^2 u_j + f(u_j) & \text{on the edges } k_j, \\ (K) \quad \sum_j d_{ij} c_{ij} \partial_j u_j(v_i, t) = 0 & \text{at the vertices } v_i. \end{cases}
$$

Besides some general exclusion criteria based on the spectral properties of the linearized differential operators, it will be shown that there are no stable stationary non constant solutions of the evolution problem (1). The results presented here stem from a joint work with José Antonio Lubary, UPC Barcelona, Spain.

## SCATTERING FOR HAMILTONIAN SYSTEMS

Christer Bennewitz
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I will describe how the so called Conservative Camassa-Holm system, of importance to the theory of water waves, leads to a scattering/inverse scattering problem for a $2 \times 2$ Hamiltonian system. I will then sketch a general scattering theory for such systems, which also involves a spectral transform we call the Jost transform. Finally, I will indicate the proof of a uniqueness theorem for inverse scattering for equations of this type, which is also a uniqueness theorem for inverse spectral theory.
This is joint work with Malcolm Brown in Cardiff and Rudi Weikard in Birmingham, AL.

## ON PÓLYA'S INEQUALITY FOR TORSIONAL RIGIDITY AND FIRST DIRICHLET EIGENVALUE

Michiel van den Berg
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Let $\Omega$ be an open set in Euclidean space with finite Lebesgue measure $|\Omega|$. We obtain some properties of the set function $F : \Omega \mapsto \mathbb{R}^+$ defined by

$$F(\Omega) = \frac{T(\Omega)\lambda_1(\Omega)}{|\Omega|},$$

where $T(\Omega)$ and $\lambda_1(\Omega)$ are the torsional rigidity and first eigenvalue of the Dirichlet Laplacian respectively. We improve the classical Pólya bound

$$F(\Omega) \leq 1,$$

and show that for convex sets in $\mathbb{R}^2$,

$$F(\Omega) \leq 1 - \frac{1}{13000}.$$

For any $\epsilon \in (0, 1)$ we construct an open set $\Omega_\epsilon$ such that $F(\Omega_\epsilon) \geq 1 - \epsilon$. This is joint work with V. Ferone, C. Nitsch, C. Trombetti.

## AN INVERSE SCATTERING PROBLEM IN RANDOM MEDIA

Pedro Caro
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In inverse scattering theory the aim is to determine a scattering potential $q$ from appropriate measurements. In many applications the scatterer is non-smooth and vastly complicated. For such scatterers, the inverse problem is not so much to recover the exact micro-structure of an object but merely to determine the parameters or functions describing the properties of the micro-structure. An example of such a parameter is the correlation length of the medium which is related to the typical size of "particles" inside the scatterer. In mathematical terms, the potential $q$ is assumed to be a Gaussian random function whose covariance operator is a classical pseudo-differential operator. We show that the backscattered field, obtained from a single realization of the random potential $q$, determines uniquely the principal symbol of the covariance operator. This is a joint work with Tapio Helin and Matti Lassas.

## BOUNDARY TRIPLES, KREIN FORMULA AND RESOLVENT ESTIMATES FOR ONE-DIMENSIONAL HIGH-CONTRAST PERIODIC PROBLEMS

Kirill Cherednichenko
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I will discuss operator-norm resolvent convergence estimates for one-dimensional $\varepsilon$-periodic differential operators $A_\varepsilon$ with rapidly oscillating coefficients in the non-uniformly elliptic high-contrast $(1 : \varepsilon^2)$ setting, which has been out of reach of the existing homogenisation techniques. Our analysis is based on a special representation of the resolvent of the fibres of $A_\varepsilon$ in terms of the $M$-matrix of an associated boundary triple, due to M. G. Krein. The resulting asymptotic behaviour as $\varepsilon \to 0$, is shown to be described, up to a unitary equivalence, by a non-standard version of the Kronig-Penney model on the real axis $\mathbb{R}$. This is joint work with Alexander V. Kiselev.

## ON THE GAPS IN THE SPECTRUM OF THE PERIODIC MAXWELL OPERATOR

Shane Cooper
Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK

In this talk we discuss the propagation of electromagnetic waves in a photonic crystal fibre. We demonstrate the existence of gaps in the spectrum of the periodic Maxwell operator with mildly contrasting coefficients and study the dependence on the location of these gaps with respect to the geometry of the media.

This is joint work with I. Kamotski (UCL) and V. Smyshlyaev (UCL)

## THE INVERSE SPECTRAL TRANSFORM FOR THE CONSERVATIVE CAMASSA–HOLM FLOW

Jonathan Eckhardt
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We solve the Cauchy problem for the Camassa–Holm equation (a nonlinear PDE which models shallow water waves) with decaying initial data by means of establishing the corresponding inverse spectral transform, that is, solving an inverse spectral problem for an indefinite Sturm–Liouville problem of the form

$$-f'' + \frac{1}{4}f = z\,\omega f + z^2 \upsilon f.$$

In particular, the conservative Camassa–Holm flow is completely linearized by suitably chosen spectral quantities.

## THE RELLICH INEQUALITY IN $L^P(\Omega)$

WD Evans
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I shall report on recent joint work with David Edmunds on $L^p$ analogues of the classical Rellich inequality in general open subsets $\Omega$ of $\mathbb{R}^n$ and similar inequalities involving the $p$-Laplacian. I shall also examine the application of these inequalities to determining lower bounds on the first eigenvalues of Dirichlet problems on $\Omega$ for biharmonic and $p$-biharmonic operators

## WHAT IS THE SOLVABILITY COMPLEXITY INDEX (SCI) OF YOUR PROBLEM? - ON THE SCI HIERARCHY AND THE FOUNDATIONS OF COMPUTATIONAL MATHEMATICS

Anders Hansen
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This talk addresses some of the fundamental barriers in the theory of computations. Many computational problems can be solved as follows: a sequence of approximations is created by an algorithm, and the solution to the problem is the limit of this sequence (think about computing eigenvalues of a matrix for example). However, as we demonstrate, for several basic problems in computations such as computing spectra of operators, solutions to inverse problems, roots of polynomials using rational maps, solutions to convex optimization problems, imaging problems etc. such a procedure based on one limit is impossible. Yet, one can compute solutions to these problems, but only by using several limits. This may come as a surprise, however, this touches onto the boundaries of computational mathematics. To analyze this phenomenon we use the Solvability Complexity Index (SCI). The SCI is the smallest number of limits needed in order to compute a desired quantity. The SCI phenomenon is independent of the axiomatic setup and hence any theory aiming at establishing the foundations of computational mathematics will have to include the so called SCI Hierarchy. We will specifically discuss the vast amount of classification problems in this non-collapsing complexity/computability hierarchy that occur in inverse problems, compressed sensing problems, l1 and TV optimization problems, spectral problems, PDEs and computational mathematics in general.

## STABILITY CRITERIA FOR PERIODIC WAVES IN HAMILTONIAN SYSTEMS

Mariana Haragus
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We present three general stability/instability criteria for periodic travelling waves in Hamiltonian or reversible systems. The criteria rely upon spectral properties of the linear operators found by linearising the system at a given periodic wave. We apply these results to different model equations arising in the classical water-wave problem. The focus is on the questions of transverse and modulational linear stability/instability.

## DISCRETE DIFFRACTION MANAGED SOLITONS: THRESHOLD PHENOMENA AND RAPID DECAY FOR GENERAL NONLINEARITIES

Dirk Hundertmark
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We study solitary breather type solutions for a discrete nonlinear Schrödinger type equation. This equation is known to model a wide range of effects ranging from molecular crystals to biophysical systems. Our interest for this equation comes from the fact that it models an array of nonlinear waveguides, the hopping terms corresponding to the interaction of neighbouring waveguides. We consider a variant of this model, the diffraction management equation, where the diffraction is periodically modulated along the waveguides. In experiments stable low power pulses in these waveguide arrays have been observed.
We prove a threshold phenomenon for the existence/non-existence of energy minimizing solitary solutions of the diffraction management equation for strictly positive and zero average diffraction. Our methods allow for a large class of nonlinearities, they are, for example, allowed to change sign, and we impose the weakest possible conditions on the local diffraction profile, it only has to be locally integrable. The solutions are found as minimizers of a nonlinear and nonlocal variational problem which is translation invariant. There exists a critical threshold such that minimizers for this variational problem exist if their power is bigger than the critical threshold and no non-trivial minimizers exist with powers below the critical threshold. We also give simple criteria for the finiteness and strict positivity of the critical threshold. Our proof of existence of minimizers is rather direct and avoids the use of Lions' concentration compactness argument.
Furthermore, we give precise quantitative lower bounds on the exponential decay rate of the diffraction management solitons, which confirm the physical heuristic prediction for the asymptotic decay rate. Moreover, for ground state solutions, these bounds give a quantitative lower bound for the divergence of the exponential decay rate in the limit of vanishing average diffraction. For zero average diffraction, we prove quantitative bounds which show that the solitons decay much faster than exponential. Our results considerably extend and strengthen previous results.
Joint work with: Mi-Ran Choi and Young-Ran Lee, Sogang University, Seoul

## THE ASSOCIATED LEGENDRE EQUATION AND GENERALIZED NEVANLINNA FUNCTIONS

Heinz Langer
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We consider the equation

$$-\big((1-x^2)y'(x)\big)' + \left(\frac{1}{4} + \frac{\ell^2}{1-x^2}\right) y(x) = \lambda\, y(x), \quad x \in (-1,1),$$

and study the Titchmarsh–Weyl function, which corresponds to the Frobenius solutions at one endpoint. It is a generalized Nevanlinna function which contains all the spectral information of the problem. Joint work with C. Fulton and A. Luger.

## PATH-LAPLACIAN OPERATORS ON DISCRETE GRAPHS

Matthias Langer
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Path-Laplacian operators on discrete graphs are generalisations of discrete Laplacians where not only the neighbouring vertices are taken into account but also vertices with a larger distance. In this talk properties of these operators and certain combinations of them are presented. Moreover, a corresponding diffusion equation is discussed and when normal or anomalous diffusion occurs.

## THE WELL ORDER RECONSTRUCTION SOLUTION FOR NEMATIC LIQUID CRYSTALS

Apala Majumdar
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We analyse a new well order reconstruction solution in the Landau-de Gennes theory for nematic liquid crystals in terms of a saddle-type critical point of an associated scalar variational problem. This solution was reported numerically in three-dimensional square wells, by Kralj&Majumdar in 2014. We prove the existence of this well order reconstruction solution, study its qualitative properties and prove that the solution loses stability for large wells, bifurcating into the more familiar diagonal and rotated solutions for large wells. In particular, we prove that this order reconstruction solution undergoes a supercritical pitchfork bifurcation in the scalar setting by a careful study of the second variation of the scalar energy.This is joint work with Samo Kralj, Giacomo Canevari, Amy Spicer, Martin Robinson, Chong Luo and Radek Erban.

## INVERSE PROBLEMS FOR THE CONNECTION LAPLACIAN

Lauri Oksanen
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We reconstruct a Riemannian manifold and a Hermitian vector bundle with compatible connection from the hyperbolic Dirichlet-to-Neumann operator associated to the wave equation for the connection Laplacian. The boundary data is local and the reconstruction is up to the natural gauge transformations of the problem. The gauge transformations are present already in the scalar case where the connection corresponds to a magnetic vector potential. The hyperbolic inverse problem is closely related to the corresponding inverse boundary spectral problem. The talk is based on joint work Yaroslav Kurylev (University College London) and Gabriel P. Paternain (University of Cambridge).

## EFFECTIVE HAMILTONIANS FOR SPECTRAL ASYMPTOTICS OF ROBIN LAPLACIANS

Konstantin Pankrashkin
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For an open set $\Omega \subset \mathbf{R}^d$ with a suitably regular boundary and a constant $\alpha > 0$, denote by $Q_\alpha^\Omega$ the Laplacian $u \mapsto -\Delta u$ acting in $L^2(\Omega)$ on the functions $u$ satisfying the Robin boundary conditions $D_n u = \alpha u$ on $\partial\Omega$, where $D_n$ is the outer unit normal derivative.
On the other hand, denote by $M$ be the sum of the principal curvatures, which is defined on $\partial\Omega$, and consider the Schrödinger operator

$$L_\alpha^\Omega := -\Delta_{\partial\Omega} - \alpha M$$

acting in $L^2(\partial\Omega)$, where $\Delta_{\partial\Omega}$ is the Laplace-Beltrami operator.
We show that in various situations there is a link between the spectral properties of $Q_\alpha^\Omega$ and those of $L_\alpha^\Omega$. In particular, we discuss the eigenvalue asymptotics for $\alpha \to +\infty$ and the behavior of the discrete spectrum on conical domains.
The talk is based on joint works with Vincent Bruneau and Nicolas Popoff (Bordeaux).

## THRESHOLDS IN THE SPECTRUM OF HAMILTONIAN WITH TRANSLATION INVARIANT MAGNETIC FIELDS

Nicolas Popoff
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The goal of this talk is to present different models with translation invariant magnetic field, such as Iwatsuka models. The associated Hamiltonians can be fibered, and the properties of the band functions can be linked to the propagation properties of quantum states submitted to such magnetic fields. If the case of states with bounded frequencies is well understood, less is known for the states with energy at a threshold corresponding to a limit of a band function. Asymptotics of the band functions for large frequencies allows to have a better description of these states, and is a necessary step for the description of the resolvent near the thresholds. I will also present in particular cases results concerning the number of eigenvalues below the essential spectrum for suitable perturbations of such operators.

## WILDLY PERTURBED MANIFOLDS: NORM RESOLVENT AND SPECTRAL CONVERGENCE

Olaf Post
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Wildly perturbed manifolds refers to manifolds where an increasing number of smaller and smaller holes is removed. We consider Neumann and Dirichlet Laplacians and show a norm resolvent convergence to some Laplace operator in the limit. The limit operator depends on the number of holes removed. If the removed holes remain sparse, the limit operator is the original Laplacian on the manifold; if the removed holes 'solidify', the effect is seen in the limit operator. The problem is related to the famous 'crushed ice' problem of Rauch and Taylor. We also consider manifolds with an increasing number of handles attached and discuss the limit operator. (Joint work with Colette Anné, Nantes)

## RATIONAL APPROXIMATION OF FUNCTIONS WITH LOGARITHMIC SINGULARITIES

Alexander Pushnitski
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I will report on the results of my recent work with Dmitri Yafaev (Rennes-1). We consider functions $\omega$ on the unit circle with a finite number of logarithmic singularities. We study the approximation of $\omega$ by rational functions in the BMO norm. We find the leading term of the asymptotics of the distance in the BMO norm between $\omega$ and the set of rational functions of degree $n$ as $n$ goes to infinity. Our approach relies on the Adamyan-Arov-Krein theorem and on the study of the asymptotic behaviour of singular values of Hankel operators. In particular, we make use of the localisation principle, which allows us to combine the contributions of several singularities into a single asymptotic formula.

## ORDINARY DIFFERENTIAL OPERATORS WITH SINGULAR COEFFICIENTS

Andrei Shkalikov
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Classical theory of symmetric differential operators allows to work with differential expressions of the form

$$(1)\quad \ell_{2n}[y] := \sum_{k=0}^{n}(-1)^{n-k}(p_k\, y^{(n-k)})^{(n-k)} + \\ + i\sum_{k=0}^{n-1}(-1)^{n-k-1}\{(q_k\, y^{(n-k-1)})^{(n-k)} + (q_k\, y^{(n-k)})^{(n-k-1)}\},$$

provided that the coefficients $p_k$, $q_k$ and $(p_0)^{-1}$ are locally integrable functions. This theory was developed in the works of D.Shin, N.Glazman, A.Zettle, N.Everitt et al. Our goal is to extend the frames of the classical theory and to define operators associated with non-symmetric differential expressions

$$(2)\qquad \tau(y) = \sum_{k,s=0}^{n}\left(r_{ks}y^{(n-k)}\right)^{(n-s)}$$

whose coefficients $r_{ks}$ are distributions of finite order singularity (depending on the indices $k, s$). We shall discuss several approaches to this problem. The most important one is based on the so-called regularization procedure.
The talk is based on the joint paper with prof. K.A.Mirzoev.

## SPECTRAL BOUNDS FOR NON-SELFADJOINT OPERATORS AND OPERATOR MATRICES

Christiane Tretter
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In this talk methods to enclose the spectra of certain classes of non-selfadjoint operators will be presented. Special attention will be paid to operator matrices and to classes of operators that appear frequently in applications.

# CONSTRUCTION OF THE SELFADJOINT DILATION OF A MAXIMAL DISSIPATIVE OPERATOR - AN APPROACH IN THE SPIRIT OF BOUNDARY TRIPLES

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Similarly to the way that boundary triples naturally associate ‘boundary operators’ with an adjoint pair of operators, allowing e.g. Weyl functions to be introduced in an abstract setting, we introduce an abstract framework for a maximally dissipative operator $A$ and its anti-dissipative adjoint.
In the framework, we construct the selfadjoint dilation of $A$ using the Štraus characteristic function. The advantage of this construction is that the parameters arising in the dilation are explicitly given in terms of parameters of $A$ (such as coefficients of a differential expression).
The abstract theory will be illustrated by the example of dissipative Schrödinger operators.